Simple instrument rules, optimal simple instrument rules, and certainty equivalence

- Model on state-space form

$$\begin{bmatrix} X_{t+1} \\ Hx_{t+1|t} \end{bmatrix} = A \begin{bmatrix} X_t \\ x_t \end{bmatrix} + Bi_t + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{1}$$

- Instrument rule (linear)

$$i_t = f \begin{bmatrix} X_t \\ x_t \end{bmatrix} \equiv f_X X_t + f_x x_t$$

$f_x \equiv 0$: Explicit instrument rule (responds to predetermined variables only)
$f_x \neq 0$: Implicit instrument rule (responds to forward-looking variables)

– Simultaneity, equilibrium condition

– Implementation problem

- Simple instrument rule: Few arguments

– Taylor rule

$$i_t = r + \pi^* + 1.5(\pi_t - \pi^*) + 0.5(y_t - \bar{y}_t)$$

- Equilibrium system with instrument rule

$$\begin{bmatrix} X_{t+1} \\ Hx_{t+1|t} \end{bmatrix} = (A + Bf) \begin{bmatrix} X_t \\ x_t \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1}$$

ε_t has zero mean and covariance I_{n_ε}

- Solution (assume unique bounded, condition on A, B, f)

$$x_t = G(f) X_t$$

$$X_{t+1} = M(f) X_t + C\varepsilon_{t+1}$$

$$M(f) \equiv A_{11} + A_{12} G(f) + B_1 f \begin{bmatrix} I \\ G(f) \end{bmatrix}$$

• Unconditional second moments (unconditional variance if $\mathrm{E}[X_t] = 0$, no constants)

$$\mathrm{E}[X_t X_t'|f] = M(f)\mathrm{E}[X_t X_t'|f]M(f)' + CC'$$

• Intertemporal loss ($0 < \delta < 1$)

$$\mathrm{E}_t \sum_{\tau=0}^{\infty} (1-\delta)\delta^\tau L_{t+\tau}$$

$$L_t \equiv \frac{1}{2} \begin{bmatrix} X_t \\ x_t \end{bmatrix}' W \begin{bmatrix} X_t \\ x_t \end{bmatrix} = \frac{1}{2} X_t' \begin{bmatrix} I \\ G(f) \end{bmatrix}' W \begin{bmatrix} I \\ G(f) \end{bmatrix} X_t \equiv \frac{1}{2} X_t' \bar{W}(f) X_t$$

• Equilibrium intertemporal loss
Conditional on X_t

$$\frac{1}{2}[(1-\delta)X_t'V(f)X_t + \delta w(f)]$$

$$V(f) = \bar{W}(f) + \delta M(f)'V(f)M(f)$$

$$w(f) = \mathrm{tr}(VCC')$$

Unconditional equilibrium loss

$$\mathrm{E}[\sum_{\tau=0}^{\infty}(1-\delta)\delta^\tau L_{t+\tau}|f] \;=\; \mathrm{E}[L_t|f] \;=\; \frac{1}{2}\mathrm{E}[X_t'\bar{W}(f)X_t|f] \;=\; \frac{1}{2}\mathrm{E}\{\mathrm{tr}[X_t'\bar{W}(f)X_t]|f\}$$

$$= \frac{1}{2}\mathrm{tr}\{\bar{W}(f)\mathrm{E}[X_t X_t'|f]\}$$

$$\frac{1}{2}\mathrm{E}[(1-\delta)X_t'V(f)X_t + \delta w(f)|f] = \frac{1}{2}\left\{(1-\delta)\mathrm{tr}\{V(f)\mathrm{E}[X_t X_t'|f]\} + \delta w(f)\right\}$$

• Optimal simple instrument rule
Consider class $\mathcal{F} \subset R^{n_X + n_x}$ of simple instrument rules, $f \in \mathcal{F}$
Consider commitment in period t to optimal simple instrument rule

– Conditional intertemporal loss

$$f(X_t) = \arg\min_{f \in \mathcal{F}} \frac{1}{2}[(1-\delta)X_t'V(f)X_t + \delta w(f)]$$

Optimal simple rule depends on X_t (Currie-Levine)

– Unconditional intertemporal loss

$$f = \arg\min_{f \in \mathcal{F}} \frac{1}{2}\mathrm{E}[L_t|f] = \frac{1}{2}\mathrm{tr}\{\bar{W}(f)\mathrm{E}[X_t X_t'|f]\}$$

• For simple instrument rules, no certainty equivalence