The Kalman filter

(Hamilton 94, Harvey 81, 89, Ljungqvist-Sargent 04 (sect. 5.6, app. B))
X_t unobservable variables, Z_t observable variables
Measurement equation:

$$Z_t = LX_t + v_t,$$

where measurement errors v_t is $N(0, \Sigma_{vv})$.
Transition equation:

$$X_{t+1} = TX_t + u_{t+1},$$

where shocks u_t is $N(0, \Sigma_{uu})$ and $\mathrm{E}[u_t v_s'] = 0$ for all t and s.
Prediction/estimation errors: $X_t - X_{t|t-1}$, $X_t - X_{t|t}$.
Covariances matrices of prediction/estimation errors:

$$P_{t|t-1} \equiv \mathrm{E}[(X_t - X_{t|t-1})(X_t - X_{t|t-1})']$$
$$P_{t|t} \equiv \mathrm{E}[(X_t - X_{t|t})(X_t - X_{t|t})'].$$

(The matrix $P_{t|t}^{-1}$ is called the *information* matrix.)

Innovations:

$$Z_t - Z_{t|t-1} = L(X_t - X_{t|t-1}) + v_t.$$

The covariance matrix of the innovations:

$$\mathrm{E}[(Z_t - Z_{t|t-1})(Z_t - Z_{t|t-1})'] = LP_{t|t-1}L' + \Sigma_{vv}.$$

Prediction equations:

$$X_{t|t-1} = TX_{t-1|t-1},$$

$$P_{t|t-1} = TP_{t-1|t-1}T' + \Sigma_{uu}.$$

Updating equations:
For $X_{t|t}$:

$$X_{t|t} = X_{t|t-1} + K_t(Z_t - LX_{t|t-1}),$$

where the Kalman gain matrix, K_t, is given by

$$K_t \equiv P_{t|t-1}L'(LP_{t|t-1}L' + \Sigma_{vv})^{-1}.$$

For $P_{t|t}$:

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}L'(LP_{t|t-1}L' + \Sigma_{vv})^{-1}LP_{t|t-1}.$$

Note that the estimate of the meaurement error is given by

$$v_{t|t} = Z_t - Z_{t|t-1} - L(X_{t|t} - X_{t|t-1}) = (I - LK_t)(Z_t - Z_{t|t-1})$$

In steady state,

$$\begin{aligned}
P_{t|t-1} &= P, \\
P_{t|t} &= P - PL'(LPL' + \Sigma_{vv})^{-1}LP, \\
K_t &= K, \\
K &= PL'(LPL' + \Sigma_{vv})^{-1}, && (1) \\
P &= T[P - PL'(LPL' + \Sigma_{vv})^{-1}LP]T' + \Sigma_{uu}. && (2)
\end{aligned}$$

Note similarity between (1) and (2) and the equtions for optimal policy function F and value function matrix V for LQR. Equations for P and V are both Riccati equtations. See Ljungqvist-Sargent (04 App. B) on more precise duality between LQR and Kalman filter.

Optimal monetary policy with partial information

Indicator Variables for Optimal Policy

Svensson-Woodford 2003

- Monetary policy under uncertainty, partial information (state of economy, disturbances)
  - Inflation-forecast targeting: Conditional inflation forecasts
  - Indicators: Conflicting information
  - Optimal weights?
  - Principles known for backward-looking variables
- Purpose
  - Clarify principles for optimal weights on indicators when forward-looking variables
  - Clarify optimal policy with partial information, under commitment and discretion

- Without forward-looking variables
  - Linear model, quadratic loss function
  - Certainty-equivalence
  - Separation principle
    - Optimization (optimal linear regulator)
    - Estimation (Kalman filter)
  - Chow 1975, Kalchenbrenner-Tinsley 1975, LeRoy-Waud 1977
  - Restrictive
    - Many indicators are forward-looking: asset prices, private-sector inflation expectations, order-flows, confidence measures, etc.

- With forward-looking variables
  - Estimation problem more complex: Circularity
    Estimates which depend on indicators, which depend on estimates
  - Discretion different from commitment
    (Backus-Driffill, Currie-Levine)
  - Estimation, linear non-optimizing model: Complex Kalman filter
    (Pearlman-Currie-Levin 1986)
  - Optimization: Certainty-equivalence, separation principle (Pearlman 1992)
- Svensson-Woodford 03
  - Much simpler derivation of optimal weights
  - Handle circularity in operational way
  - Example: Unobservable potential output (Orphanides 2003b)
  - Symmetric info (asymmetric info: Svensson-Woodford 04)

• Outline

2 Optimization under discretion

– Model, discretion, certainty-equivalence, Kalman filter

3 Optimal policy with commitment

4 Optimal weights on indicators: General remarks

– Handling the circularity

5 Example: Monetary policy with unobservable potential output

6 Conclusions

Appendices A, B

2 Optimization under discretion

Model: Private sector, central bank

$$\begin{bmatrix} X_{t+1} \\ \Omega x_{t+1|t} \end{bmatrix} = A^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + A^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + Bi_t + \begin{bmatrix} u_{t+1} \\ 0 \end{bmatrix} \quad (1)$$

X_t predetermined variables, x_t forward-looking variables, i_t instrument(s), u_t iid shocks

$z_{\tau|t} \equiv \mathrm{E}[z_\tau | I_t]$ estimate in period t

Target variables

$$Y_t = C^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + C^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + C_i i_t \quad (2)$$

Period loss function

$$L_t = Y_t' W Y_t \quad (3)$$

Observable variables (indicators)

$$Z_t = D^1 \begin{bmatrix} X_t \\ x_t \end{bmatrix} + D^2 \begin{bmatrix} X_{t|t} \\ x_{t|t} \end{bmatrix} + v_t \quad (4)$$

Information in period t

$$I_t = \{Z_\tau, \tau \le t;\ A^1, A^2, B, C^1, C^2, C_i, D^1, D^2, \Omega, W, \delta, \Sigma_{uu}, \Sigma_{vv}\} \quad (5)$$

Discretion: No commitment mechanism
Intertemporal loss function

$$\mathrm{E}[\sum_{\tau=0}^{\infty} \delta^{\tau} L_{t+\tau} | I_t] \tag{6}$$

- Certainty equivalence (appendix A, Pearlman 92)
  Assume

$$\begin{aligned} i_{t+1} &= F_{t+1} X_{t+1|t+1} \\ x_{t+1|t+1} &= G_{t+1} X_{t+1|t+1} \end{aligned}$$

  Derive

$$\begin{aligned} x_{t|t} &= \tilde{A}_t X_{t|t} + \tilde{B}_t i_t \\ X_{t+1|t} &= A_t^* X_{t|t} + B_t^* i_t \end{aligned}$$

$$L_{t|t} = \begin{bmatrix} X_{t|t} \\ i_t \end{bmatrix}' \begin{bmatrix} Q_t^* & U_t^* \\ U_t^{*\prime} & R_t^* \end{bmatrix} \begin{bmatrix} X_{t|t} \\ i_t \end{bmatrix} + l_t$$

  Bellman equation

$$X_{t|t}' V_t X_{t|t} + w_t = \min_{i_t} \left\{ L_{t|t} + \delta \mathrm{E}[X_{t+1|t+1}' V_{t+1} X_{t+1|t+1} + w_{t+1}] \right\}$$

  Results in

$$\begin{aligned} i_t &= F_t X_{t|t} \\ x_t &= G_t X_{t|t} \end{aligned}$$

  Mapping $(V_{t+1}, F_{t+1}, G_{t+1}) \longmapsto (V_t, F_t, G_t)$, fixpoint (V, F, G)

Implied reaction function

$$i_t = FX_{t|t} \tag{7}$$

Estimate of forward-looking variables

$$x_{t|t} = GX_{t|t} \tag{8}$$

Forward-looking variables ($G = G^1 + G^2$, see paper for details)

$$x_t = G^1 X_t + G^2 X_{t|t} \tag{12}$$

- F, G^1 and G^2 independent of D^1, D^2, Σ_{uu} and Σ_{vv}
  - Independent of information structure
  - Independent of variance of shocks
  - Policy same as under full information
  - Certainty equivalence

Dynamics for X_t and Z_t

$$\begin{aligned} X_{t+1} &= HX_t + JX_{t|t} + u_{t+1} & (15) \\ Z_t &= LX_t + MX_{t|t} + v_t & (16) \end{aligned}$$

where

$$\begin{aligned} H &\equiv A^1_{11} + A^1_{12} G^1 & (17) \\ J &\equiv B_1 F + A^1_{12} G^2 + A^2_{11} + A^2_{12} G & (18) \\ L &\equiv D^1_1 + D^1_2 G^1 & (19) \\ M &\equiv D^1_2 G^2 + D^2_1 + D^2_2 G & (20) \end{aligned}$$

- Estimation problem without forward-looking variables!
  - (15) transition equation, with $X_{t|t}$
  - (16) measurement equation, with $X_{t|t}$
- Much simpler than Pearlman-Currie-Levine 1986

2.1 Optimal filtering

Assume updating equation (K Kalman gain)

$$X_{t|t} = X_{t|t-1} + K(Z_t - LX_{t|t-1} - MX_{t|t}) \tag{21}$$

Rationalize

$$Z_t - LX_{t|t-1} - MX_{t|t} = L(X_t - X_{t|t-1}) + v_t$$

Conventional Kalman updating equation

$$X_{t|t} = X_{t|t-1} + K[L(X_t - X_{t|t-1}) + v_t] \tag{22}$$

Prediction equation, from (15)

$$X_{t+1|t} = (H + J)X_{t|t} \tag{23}$$

- Dynamics: (15), (12), (22) and (23)

Remains to determine K (appendix B)

$$K = PL'(LPL' + \Sigma_{vv})^{-1} \tag{24}$$

$P \equiv \mathrm{Cov}[X_t - X_{t|t-1}]$ covariance of prediction errors

$$P = H[P - PL'(LPL' + \Sigma_{vv})^{-1}LP]H' + \Sigma_{uu} \tag{25}$$

Solve P from (25), numerically or analytically (Riccati equation)

- K independent of policy (depends on A^1, D^1, Σ_{uu} and Σ_{vv}, *not* on C^1, C^2, C_i, W and δ)
  Separation principle (as w/o forward-looking variables)
  Not under asymmetric info (Svensson-Woodford 04)

- Determine K (appendix B)

$$\tilde{X}_t \equiv X_t - X_{t|t-1}$$
$$\tilde{Z}_t \equiv Z_t - Z_{t|t-1}$$

Measurement equation

$$\tilde{Z}_t = N\tilde{X}_t + \nu_t$$

$$N \equiv (I + MK)L$$
$$\nu_t \equiv (I + MK)v_t$$

Transition equation

$$\tilde{X}_{t+1} = T\tilde{X}_t + \omega_{t+1}$$

$$T \equiv H(I - KL)$$
$$\omega_{t+1} \equiv u_{t+1} - HKv_t$$

Standard (stationary) Kalman filter

Updating equation ($\tilde{X}_{t|t-1} = 0$, $\tilde{Z}_{t|t-1} = 0$)

$$\tilde{X}_{t|t} = \tilde{K}(N\tilde{X} + \nu_t) \tag{B.9}$$

Kalman gain

$$\tilde{K} \equiv PN'(NPN' + \Sigma_{\nu\nu})^{-1}$$

$$P = TPT' + \Sigma_{\omega\omega}$$
$$\Sigma_{\nu\nu} \equiv (I + MK)\Sigma_{vv}(I + MK)'$$
$$\Sigma_{\omega\omega} = HK\Sigma_{vv}K'H' + \Sigma_{uu}$$

Use (21), solve for $X_{t|t}$, express in terms of $\tilde{Z}_t$

$$X_{t|t} = X_{t|t-1} + K(I + MK)^{-1}\tilde{Z}_t \tag{B.13}$$

(used $(I + KM)^{-1} \equiv I - (I + KM)^{-1}KM$, $(I + KM)^{-1}K \equiv K(I + MK)^{-1}$)
Comparing (B.9) and (B.13) gives

$$K(I + MK)^{-1} = PN'(NPN' + \Sigma_{\nu\nu})^{-1}$$

Substitution gives (24)

3 Optimal policy with commitment

Still certainty-equivalence (Pearlman 92, Svensson-Woodford 02)

$$i_t = FX_{t|t} + \Phi\,\Xi_{t-1} \tag{26}$$

$$x_{t|t} = GX_{t|t} + \Gamma\,\Xi_{t-1} \tag{27}$$

$$\Xi_t = SX_{t|t} + \Sigma\,\Xi_{t-1} \tag{28}$$

Ξ_t Lagrange multiplier of the forward-looking variables, $\Xi_{t_0-1} = 0$.
"Timeless perspective", stationary equilibrium, social optimum
(Woodford 03, Svensson-Woodford 05) $t_0 \to -\infty$.

$$\Xi_{t-1} = \sum_{\tau=0}^{\infty} \Sigma^{\tau} SX_{t-1-\tau|t-1-\tau}$$

- Inertia, history-dependence
  (Woodford 03, Svensson-Woodford 05).
- Achieve optimal policy under discretion, modify current loss function, "commitment to continuity and predictability," (Svensson-Woodford 05), RSP method

$$\mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^{\tau} L_{t+\tau} + \Xi_{t-1}(x_t - x_{t|t-1}) \tag{29}$$

Dynamics under commitment

$$x_t = G^1 X_t + G^2 X_{t|t} + \Gamma\,\Xi_{t-1} \tag{30}$$

$$X_{t+1} = HX_t + JX_{t|t} + \Psi\,\Xi_{t-1} + u_{t+1} \tag{31}$$

$$\Psi \equiv A_{12}\Gamma + B_1\Phi \tag{32}$$

3.1 Optimal filtering

$$Z_t = LX_t + MX_{t|t} + \Lambda\,\Xi_{t-1} + v_t \tag{33}$$

$$\Lambda \equiv D_2\Gamma \tag{34}$$

Updating equation

$$X_{t|t} = X_{t|t-1} + K(Z_t - LX_{t|t-1} - MX_{t|t} - \Lambda\,\Xi_{t-1}) \tag{35}$$

Prediction equation

$$X_{t+1|t} = (H + J)X_{t|t} + \Psi\,\Xi_{t-1} \tag{36}$$

- Just add dependence on Ξ_{t-1}

4 Optimal weights on indicators: General remarks
(Discretion case, $\Phi = 0$, $\Gamma = 0$, $S = 0$, $\Sigma = 0$, $\Psi = 0$, $\Lambda = 0$, $\Xi_{t-1} \equiv 0$)
Observable variables

$$\begin{aligned} Z_t &= D_1^1 X_t + D_2^1 x_t + D_1^2 X_{t|t} + D_2^2 x_{t|t} + v_t \\ &= L X_t + M X_{t|t} + v_t \end{aligned}$$

- $M \neq 0$, circularity

"Ideal" indicators

$$\bar{Z}_t \equiv Z_t - M X_{t|t} \tag{37'}$$

$$\bar{Z}_t = L X_t + v_t$$

Innovation

$$\bar{Z}_t - \bar{Z}_{t|t-1} = L(X_t - X_{t|t-1}) + v_t$$

- Not operational

Solve for $X_{t|t}$ in (21)

$$X_{t|t} = (I + KM)^{-1}[(I - KL)X_{t|t-1} + KZ_t] \tag{38'}$$

- Operational recursive updating equation

Dynamics for $X_{t|t}$

$$X_{t|t} = (I + KM)^{-1}\{[(I - KL)(H + J)]X_{t-1|t-1} + KZ_t\}$$

Solve backwards

$$X_{t|t} = \sum_{\tau=0}^{\infty} Q_\tau (I + KM)^{-1} K Z_{t-\tau}$$

$$Q_\tau \equiv [(I + KM)^{-1}(I - KL)(H + J)]^\tau$$

- Contemporaneous effect, M
- Gradual updating, inertia, history-dependence
  Also under discretion, less inefficient?

Optimal weights

- Row j, $K_{j\cdot}$
  Weights on Z_t in updating element X_{jt}
- Column l, $K_{\cdot l}$
  Weights on element Z_{lt} in updating X_t
- Assume row j, $L_{j\cdot}$: $L_{jl} = 0$ for $l \neq j$, $L_{jj} = 1$

$$Z_{jt} = X_{jt} + M_{j\cdot} X_{t|t} + v_{jt}$$

  X_{jt} observed with measurement error v_{jt}
  No error, $\sigma^2_{vj} \to 0$: $k_{jl} \to 0$, $l \neq j$, $k_{jj} \to 1$
  $X_{jt|t} = X_{jt}$, X_{jt} observable
  Unbounded error, $\sigma^2_{\nu j} \to \infty$: Column $K_{\cdot j} = 0$
  Z_{jt} useless indicator

- Policy reaction function (respond to optimal estimate or to indicators)

$$i_t = F X_{t|t} = F \sum_{\tau=0}^{\infty} Q_\tau (I + KM)^{-1} K Z_{t-\tau}$$

  – Response to $X_{t|t}$: Certainty equivalence
  – Response to Z_t: Not certainty equivalence

Orphanides 03, "The Quest for Prosperity without Inflation"

- 70s, great inflation, too expansionary policy
- Uncertainty about potential output (estimated output gap too low in 70s, therefore too expansionary policy) (figure 2)
- Taylor type instrument rule

$$i_t = r + \pi^* + \gamma(\pi_t - \pi^*) + \delta(y_t - \bar{y}_t)$$

- Uncertainty about $\bar{y}_t$ suggests "prudence," $\delta \approx 0$

Critical questions

- Status of CEA estimates of potential output?
  - Noisy/biased estimates?
  - Optimal estimates?
  - How much used?
- Conceptual clarification
  - Response to optimal estimates (certainty equivalence?)
  - Response to noisy observations (optimal filtering)
  - Optimal instrument rule (certainty equivalence)
  - Simple instrument rule (not certainty equivalence)

5 Example: Unobservable potential output

Model of Svensson-Woodford 03

$$\pi_t = \delta\pi_{t+1|t} + \kappa(y_t - \bar{y}_t) + \nu_t \quad (41)$$

$$\bar{y}_{t+1} = \gamma\bar{y}_t + \eta_{t+1} \quad (43)$$

$$\nu_{t+1} = \rho\nu_t + \varepsilon_{t+1} \quad (44)$$

π_t inflation, y_t output (simplification: control variable), $\bar{y}_t$ potential output, ν_t "cost-push" shock; shocks ε_t, η_t iid, zero mean, variances σ_ε^2, σ_η^2

Aggregate demand curve

$$y_t = y_{t+1|t} - \sigma(i_t - \pi_{t+1|t}) \quad (42)$$

$$i_t = \pi_{t+1|t} + \frac{1}{\sigma}(y_{t+1|t} - y_t) \quad (48)$$

Period loss function: "flexible inflation targeting"

$$L_t = \frac{1}{2}[\pi_t^2 + \lambda(y_t - \bar{y}_t)^2] \quad (45)$$

Imperfect observation of potential output, shock θ_t, zero mean, variance σ_θ^2

$$\tilde{y}_t = \bar{y}_t + \theta_t, \tag{46}$$

Direct observation of inflation

Observable variables

$$Z_t = \begin{bmatrix} \bar{y}_t + \theta_t \\ \pi_t \end{bmatrix} \tag{47}$$

- Circularity: π_t depends on $\bar{y}_{t|t}$, which depends on π_t

Rewrite the model

$$\begin{bmatrix} X_t \\ \hline x_t \end{bmatrix} \equiv \begin{bmatrix} \bar{y}_t \\ \nu_t \\ \hline \pi_t \end{bmatrix}$$

$$\begin{bmatrix} \bar{y}_{t+1} \\ \nu_{t+1} \\ \hline \pi_{t+1|t} \end{bmatrix} = \left[\begin{array}{cc|c} \gamma & 0 & 0 \\ 0 & \rho & 0 \\ \hline \kappa/\delta & -1/\delta & 1/\delta \end{array}\right] \begin{bmatrix} \bar{y}_t \\ \nu_t \\ \hline \pi_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \hline -\kappa/\delta \end{bmatrix} y_t + \begin{bmatrix} \eta_{t+1} \\ \varepsilon_{t+1} \\ \hline 0 \end{bmatrix} \tag{49}$$

$$Z_t = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right] \begin{bmatrix} \bar{y}_t \\ \nu_t \\ \hline \pi_t \end{bmatrix} + \begin{bmatrix} \theta_t \\ \hline 0 \end{bmatrix}$$

5.1 Equilibrium under discretion and commitment

Discretion

"Policy function" (targeting rule)

$$y_t = \bar{y}_{t|t} - \frac{\kappa}{\kappa^2 + \lambda(1-\delta\rho)}\nu_{t|t}$$

– Response to estimate independent of uncertainty

Forward-looking variable

$$\pi_t = \pi_{t|t} = \frac{\lambda}{\kappa^2 + \lambda(1-\delta\rho)}\nu_{t|t}$$

Commitment

"Policy function" (targeting rule)

$$y_t = \bar{y}_{t|t} - \frac{\kappa}{\lambda}\frac{\mu}{1-\delta\rho\mu}\nu_{t|t} - \frac{\kappa}{\lambda}\mu\Xi_{t-1} \tag{50}$$

– Response to estimate independent of uncertainty

Forward-looking variable

$$\pi_t = \pi_{t|t} = \frac{\mu}{1-\delta\rho\mu}\nu_{t|t} - (1-\mu)\Xi_{t-1} \tag{51}$$

Lagrange multiplier

$$\Xi_t = \frac{\mu}{1-\delta\rho\mu}\nu_{t|t} + \mu\Xi_{t-1} \tag{52}$$

5.2 An optimal targeting rule

$$y_t - \bar{y}_{t|t} = -\frac{\kappa}{\lambda}\Xi_t \tag{53}$$

$$\pi_t = \Xi_t - \Xi_{t-1} \tag{54}$$

$$\pi_t = -\frac{\lambda}{\kappa}[(y_t - \bar{y}_{t|t}) - (y_{t-1} - \bar{y}_{t-1|t-1})] \tag{55}$$

– Depends only on κ and λ

– Same form as under full information (Certainty-equivalence)

Equivalent optimal targeting rule

$$p_t - p^* = -\frac{\lambda}{\kappa}(y_t - \bar{y}_{t|t}) \tag{56}$$

Note: From (54)

$$\Xi_t = p_t - p^* \tag{57}$$

5.4 The filtering problem

Rewrite solutions

$$y_t = \begin{bmatrix} 1 & f \end{bmatrix} \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} + \Phi\,(p_{t-1} - p^*), \tag{58}$$

$$\pi_t = \begin{bmatrix} 0 & g \end{bmatrix} \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} + \Gamma\,(p_{t-1} - p^*), \tag{59}$$

$$p_t = \pi_t + p_{t-1},$$

Under discretion

$$f = -\frac{\kappa}{\kappa^2 + \lambda(1 - \delta\rho)}, \quad g = -\frac{\lambda}{\kappa} f = \frac{\lambda}{\kappa^2 + \lambda(1 - \delta\rho)}$$

$$\Phi = \Gamma = \Sigma = 0.$$

Identify F and G in (7) and (8)

Under commitment

$$f \equiv -(\kappa/\lambda)[\mu/(1-\delta\rho\mu)], \quad \Phi \equiv -(\kappa/\lambda)\mu, \quad g = -\frac{\lambda}{\kappa}f = \mu/(1-\delta\rho\mu)$$

$$\Gamma = -(1-\mu), \quad \Sigma = \mu$$

Identify the matrices F, Φ, G and Γ in (26)–(27).

Forward-looking variable

$$\pi_t = \begin{bmatrix} -\kappa & 1 \end{bmatrix} \begin{bmatrix} \bar{y}_t \\ \nu_t \end{bmatrix} + \begin{bmatrix} \kappa & g-1 \end{bmatrix} \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} + \Gamma\,(p_{t-1} - p^*),$$

($\Gamma \equiv 0$ under discretion). Identification of the matrices G^1 and G^2 in (12) and (30).

Compute

$$H = \begin{bmatrix} \gamma & 0 \\ 0 & \rho \end{bmatrix}, \quad J = 0, \quad L = \begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ \kappa & g-1 \end{bmatrix}. \qquad \text{(63-64)}$$

Commitment:

$$\Psi = 0, \quad \Lambda = \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}.$$

Find 2×2 Kalman gain matrix, K, given by (24), 2×2 matrix P, given by (25). Updating equation as in section 4

$$\begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} = \begin{bmatrix} \bar{y}_{t|t-1} \\ \nu_{t|t-1} \end{bmatrix} + K(\bar{Z}_t - \bar{Z}_{t|t-1})$$

Ideal indicators

$$\bar{Z}_t = \begin{bmatrix} \bar{y}_t + \theta_t \\ \pi_t \end{bmatrix} - M \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} - \Lambda\,(p_{t-1} - p^*) = \begin{bmatrix} \bar{y}_t + \theta_t \\ -\kappa\bar{y}_t + \nu_t \end{bmatrix} \qquad \text{(65)}$$

($\Lambda \equiv 0$ under discretion).

– Observe noisy measure of $\bar{y}_t$

– Observe linear combination of $\bar{y}_t$ and ν_t

– Not operational

Solve for Kalman filter
Measurement equation (65)
Transition equation

$$\begin{bmatrix} \bar{y}_{t+1} \\ \nu_{t+1} \end{bmatrix} = H \begin{bmatrix} \bar{y}_t \\ \nu_t \end{bmatrix} + \begin{bmatrix} \eta_{t+1} \\ \varepsilon_{t+1} \end{bmatrix}, \tag{66}$$

(simple, since $\bar{y}_t$ and ν_t exogenous, $A^1_{12} = 0$, $A^2_{11} = 0$, $A^2_{12} = 0$, $B_1 = 0$)
Appendix, working paper:

$$K = \begin{bmatrix} k_{11} & k_{12} \\ \kappa k_{11} & \kappa k_{12} + 1 \end{bmatrix}. \tag{67-69}$$

$$k_{11} \equiv \frac{q}{\sigma^2_\theta}$$

(q positive root of a quadratic equation)
$\sigma^2_\theta \to 0$ implies $k_{11} \to 0$.

Operational updating equation

$$\begin{aligned} \begin{bmatrix} \bar{y}_{t|t} \\ \nu_{t|t} \end{bmatrix} &= (I + KM)^{-1} \left((I - KL) \begin{bmatrix} \bar{y}_{t|t-1} \\ \nu_{t|t-1} \end{bmatrix} - K\Lambda(p_{t-1} - p^*) + KZ_t \right) \\ &= (I + KM)^{-1} \begin{pmatrix} (I - KL)H \begin{bmatrix} \bar{y}_{t-1|t-1} \\ \nu_{t-1|t-1} \end{bmatrix} \\ -K\Lambda(p_{t-1} - p^*) + KZ_t \end{pmatrix} \end{aligned} \tag{71}$$

(simple because $J = 0$, $\Psi = 0$).

5.3 Optimal instrument rule, in terms of optimal estimates

Use (50)–(52) and (57) in (42)

$$i_t = \mu_0 (p_t - p^*) + \mu_1 (p_{t-1} - p^*) - \vartheta \bar{y}_{t|t}, \tag{62}$$

– Certainty equivalence. Coefficients independent of variances

5.5 Optimal instrument rule, in terms of observable variables

Combine (59), (62) and (71)

$$i_t = \omega i_{t-1} + \sum_{j=0}^{2} \beta_j (p_{t-j} - p^*) - \xi \, \tilde{y}_t \tag{5.34}$$

– Not certainty-equivalence. Coefficients depend on variances.

$\sigma_\theta^2 \to \infty$ implies $\xi \to 0$.

6 Conclusions

- Symmetric partial information: Certainty-equivalence, separation principle
- Inertia, history-dependence, gradual updating of estimates, also under discretion
- Circumvent circularity
- Symmetric information?
  - CB private information: Intentions (Cukierman-Meltzer, Faust-Svensson)
  - Increased transparency, more symmetric information
  - Asymmetric information: CB less info than parts of private sector (Svensson-Woodford 04)
    - ∗ Certainty equivalence: Yes (Optimal response to estimates independent of uncertainty)
    - ∗ Separation principle: No (Estimation complex, depends on policy)

- Example with unobservable potential output
  - Response to optimal estimate independent of uncertainty of estimate
  - Response to observable variables depends on information content
  - Important difference state variables/pure indicators
  - Cf. Orphanides: Are data *optimal estimates* of output gap, or *noisy observations*?

Applications

- Coenen-Levin-Wieland 05
  - Usefulness of monetary aggregates as indicators (ECB pillar, predict (cause?) inflation?)
  - Aggregate demand (output) unobservable. Money demand depends on actual aggregate demand, indicator of output
  - Potentially large information content in monetary aggregates, practically small information content
- Ehrmann-Smets 03
  - Hybrid NK model, potential output uncertain and unobservable, calibrated to euro area data
  - Loss from partial information substantial (increased variability of output gap)
  - Simple instrument rules (such as Taylor rules) that respond to optimal estimates perform relatively well
  - Under discretion, partial information reinforces the case for a "weight-conservative" CB somewhat (but small welfare gain)

- Gerali-Lippi Matlab Toolbox of Svensson-Woodford 03
  - Solution under commitment (Currie-Levine 85, Backus-Driffill 86, Ljungqvist-Sargent)
  - Solution under discretion (Oudiz-Sachs 85, Svensson-Woodford)
  - Application: NK model (Clarida-Gali-Gertler 99)