Linear-quadratic optimization with forward-looking variables: Sequential and recursive methods

Model on state-space form

$$\begin{bmatrix} X_{t+1} \\ Hx_{t+1|t} \end{bmatrix} = A \begin{bmatrix} X_t \\ x_t \end{bmatrix} + Bi_t + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{1}$$

for $t \geq 0$, X_t n_X-vector of *predetermined variables* (one element of X_t can be unity, in order to handle constants), X_0 given, x_t n_x-vector of *forward-looking variables*, i_t n_i-vector of *instruments* (control variables), ε_t n_ε-vector of exogenous zero-mean iid shocks with $\mathrm{Cov}[\varepsilon_t] = I_{n_\varepsilon}$.
Matrices A, B, C, and H are of dimension $(n_X+n_x)\times(n_X+n_x)$, $(n_X+n_x)\times n_i$, $n_X\times n_\varepsilon$, and $n_x \times n_x$, respectively.
Covariance matrix of the shocks to X_{t+1} is CC'.
$z_{t+1|t} \equiv \mathrm{E}_t z_{t+1}$.
A common special case is $H \equiv I$. In general H need not be invertible, and some rows of H may be zero.

A variable is a *predetermined* variable if and only if it has exogenous one-period-ahead forecast errors (Klein),

$$X_{t+1} - \mathrm{E}_t X_{t+1} = C\varepsilon_{t+1}.$$

Depends on lagged variables and contemporaneous exogenous shocks. (Blanchard-Kahn: *Zero* one-period-ahead forecast errors.)
Predetermined variables that only depend on lagged values of themselves and contemporaneous exogenous shocks are *exogenous* variables.
A variable that is not a predetermined variable is a *non-predetermined* variable (the forward-looking variables x_t and the instruments i_t).
Non-predetermined variables have forecast errors, $x_{t+1} - \mathrm{E}_t x_{t+1}$ and $i_{t+1} - \mathrm{E}_t i_{t+1}$, which are endogenous, that is, endogenous functions of the exogenous shocks.

$$\begin{bmatrix} X_{1,t+1} \\ X_{2,t+1} \\ [H_{11}\ H_{12}] \begin{bmatrix} x_{1,t+1|t} \\ x_{2,t+1|t} \end{bmatrix} \\ 0 \end{bmatrix} = \begin{bmatrix} A_{X11} & 0 & 0 & 0 \\ A_{X21} & A_{X22} & A_{X23} & A_{X24} \\ A_{x11} & A_{x12} & A_{x13} & A_{x14} \\ A_{x21} & A_{x22} & A_{x23} & A_{x24} \end{bmatrix} \begin{bmatrix} X_{1t} \\ X_{2t} \\ x_{1t} \\ x_{2t} \end{bmatrix} + \begin{bmatrix} 0 \\ B_{X2} \\ B_{x1} \\ B_{x2} \end{bmatrix} i_t + \begin{bmatrix} C_1 \\ C_2 \\ 0 \\ 0 \end{bmatrix} \varepsilon_{t+1}$$

The two blocks of (1) can be written

$$X_{t+1} = A_{11}X_t + A_{12}x_t + B_1 i_t + C\varepsilon_{t+1} \tag{2}$$

$$Hx_{t+1|t} = A_{21}X_t + A_{22}x_t + B_2 i_t, \tag{3}$$

where A and B are partitioned conformably with X_t and x_t,

$$A \equiv \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad B \equiv \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

The upper block, (2), determines X_{t+1} given X_t, x_t, i_t, and ε_{t+1}.
Assume that A_{22} is invertible,

$$x_t = A_{22}^{-1}(Hx_{t+1|t} - A_{21}X_t - B_2 i_t). \tag{4}$$

The lower block determines x_t given $x_{t+1|t}$, X_t, and i_t.
Assuming that the shocks ε_t only enter the upper block in (1) is not restrictive (define additional predetermined variables for these shocks and enter them in the upper block).

Expressing the constraint in the form (3) is not very restrictive. Suppose, for instance, that the constraint is replaced by a constraint of the form

$$\mathrm{E}_t \sum\nolimits_{\tau=1}^{\infty} \bar{H}^{\tau}(\bar{A}_{21}X_{t+\tau} + \bar{A}_{22}x_{t+\tau} + \bar{B}_2 i_{t+\tau}) = A_{21}X_t + A_{22}x_t + B_2 i_t, \tag{5}$$

(this form arises in some contracts problems and is a special case of the constraints in Marcet-Marimon).
First, introduce the additional forward-looking variable

$$x_{2t} \equiv \mathrm{E}_t \sum\nolimits_{\tau=0}^{\infty} \bar{H}^{\tau}(\bar{A}_{21}X_{t+\tau} + \bar{A}_{22}x_{t+\tau} + \bar{B}_2 i_{t+\tau}) = \bar{A}_{21}X_t + \bar{A}_{22}x_t + \bar{B}_2 i_t + \mathrm{E}_t \bar{H} x_{2,t+1}.$$

Second, replace (5) by the two constraints

$$\mathrm{E}_t \bar{H} x_{2,t+1} = -\bar{A}_{21}X_t - \bar{A}_{22}x_t + x_{2t} - \bar{B}_2 i_t,$$

$$0 = (A_{21} + \bar{A}_{21})X_t + (A_{22} + \bar{A}_{22})x_t - x_{2t} + (B_2 + \bar{B}_2)i_t.$$

These two constraints are obviously of the same form as (3), with

$$H \begin{bmatrix} x_{t+1|t} \\ x_{2,t+1|t} \end{bmatrix} = \begin{bmatrix} \bar{H} x_{2,t+1|t} \\ 0 \end{bmatrix}.$$

Introduce

$$Y_t \equiv D \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} \tag{6}$$

n_Y-vector of *target variables*, measured as the deviation from a fixed n_Y-vector of *target levels*, Y^*, where D is of dimension $n_Y \times (n_X + n_x + n_i)$.

Period loss function

$$L_t = \frac{1}{2} Y_t' \Lambda Y_t \equiv \frac{1}{2} \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}' W \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}, \tag{7}$$

$\Lambda, W \equiv D'\Lambda D$ symmetric positive semidefinite matrices. The elements of Λ are the weights on the target variables in the period loss function.

Intertemporal loss function in period 0 be

$$\mathrm{E}_0 \sum_{t=0}^{\infty} (1-\delta)\delta^t L_t, \tag{8}$$

where $0 < \delta < 1$ is a discount factor.

- Practice writing models on state-space form

$$y_t = \alpha(\pi_t - \pi_{t|t-1}) + \varepsilon_t$$

Optimal policy under commitment: The commitment equilibrium

Consider minimizing (8), under commitment once-and-for-all in period $t = 0$, subject to (1) for $t \geq 0$ and $X_0 = \bar{X}_0$, where $\bar{X}_0$ is given.

Variants of this problem are solved by Backus and Driffill, Currie and Levine, Sims, and Söderlind. The problem can be solved in several ways.

1. The Lagrange method (sequential method)
2. The Recursive Saddlepoint method (recursive method)

The Lagrange method: Set up the Lagrangian, derive the first-order conditions, and solve a difference equation

Rewrite (1),

$$\bar{H}\begin{bmatrix} X_{t+1} \\ x_{t+1|t} \\ i_{t+1|t} \end{bmatrix} = \bar{A}\begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix}\varepsilon_{t+1}, \tag{9}$$

$$\bar{A} \equiv \begin{bmatrix} A & B \end{bmatrix}, \qquad \bar{H} \equiv \begin{bmatrix} I & 0 & 0 \\ 0 & H & 0 \end{bmatrix}. \tag{10}$$

Lagrangian,

$$\begin{aligned}
\mathcal{L}_0 &= \mathrm{E}_0 \sum_{t=0}^{\infty}(1-\delta)\delta^t \left\{ L_t + \begin{bmatrix} \xi'_{t+1} & \Xi'_t \end{bmatrix} \left(\bar{H}\begin{bmatrix} X_{t+1} \\ x_{t+1|t} \\ i_{t+1|t} \end{bmatrix} - \bar{A}\begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} - \begin{bmatrix} C \\ 0 \end{bmatrix}\varepsilon_{t+1} \right) \right\} \\
&\quad + \frac{1-\delta}{\delta}\xi'_0(X_0 - \bar{X}_0) \\
&= \mathrm{E}_0 \sum_{t=0}^{\infty}(1-\delta)\delta^t \left\{ L_t + \begin{bmatrix} \xi'_{t+1} & \Xi'_t \end{bmatrix} \left(\bar{H}\begin{bmatrix} X_{t+1} \\ x_{t+1} \\ i_{t+1} \end{bmatrix} - \bar{A}\begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} - \begin{bmatrix} C \\ 0 \end{bmatrix}\varepsilon_{t+1} \right) \right\} \\
&\quad + \frac{1-\delta}{\delta}\xi'_0(X_0 - \bar{X}_0),
\end{aligned}$$

ξ_{t+1} and Ξ_t are vectors of n_X and n_x Lagrange multipliers of the upper and lower block, respectively, of the model equations.

Law of iterated expectations used in the second equality.

Ξ_t is dated to emphasize that it depends on information available in period t (the lower block determines x_t given information available in period t).

The FOCs with respect to X_t, x_t, and i_t for $t \geq 1$ can be written

$$\left[\, X_t' \;\; x_t' \;\; i_t' \,\right] W + \left[\, \xi_t' \;\; \Xi_{t-1}' \,\right] \frac{1}{\delta} \bar{H} - \left[\, \xi_{t+1|t}' \;\; \Xi_t' \,\right] \bar{A} = 0. \tag{11}$$

The FOCs with respect to X_t, x_t, and i_t for $t = 0$ can be written

$$\left[\, X_t' \;\; x_t' \;\; i_t' \,\right] W + \left[\, \xi_t' \;\; 0 \,\right] \frac{1}{\delta} \bar{H} - \left[\, \xi_{t+1|t}' \;\; \Xi_t' \,\right] \bar{A} = 0, \tag{12}$$

where $X_0 = \bar{X}_0$. (There is no constraint corresponding to the lower block of (9) for $t = -1$.) Define

$$\Xi_{-1} \equiv 0, \tag{13}$$

Then FOCs can be written more compactly as (11) for all $t \geq 0$ and (13).

The system of difference equations (11) has $n_X + n_x + n_i$ equations.

- The first n_X equations can be associated with the Lagrange multipliers ξ_t. The expression $-\frac{1-\delta}{\delta}\xi_t$ can be interpreted as the total marginal losses in period t of the predetermined variables X_t (for $t = 0$, with given X_0, the equations determine ξ_0). They are forward-looking variables: the Lagrange multipliers of the equations for the predetermined variables always are forward-looking, whereas the Lagrange multipliers of the equations for the forward-looking variables always are predetermined. (Substitute $X_t - \bar{X}_t$ for X_t, take derivative with respect to $\bar{X}_t$.)
- The middle n_x equations can be associated with the Lagrange multipliers Ξ_t. The expression $(1-\delta)\Xi_t' A_{22}$ can be interpreted as the total marginal losses in period t of the forward-looking variables, x_t. $(1-\delta)\Xi_t' H$ can also be interpreted as the total marginal loss in period t of the one-period-ahead expectations of the forward-looking variables, $x_{t+1|t}$. (Substitute $x_t - \bar{x}_t$ for x_t, and take derivative with respect to $\bar{x}_t$. Substitute $\bar{x}_t^e$ for $x_{t+1|t}$, and take derivative with respect to $\bar{x}_t^e$.)
- The last n_i equations are the first-order equations for the vector of instruments. In the special case when the lower right $n_i \times n_i$ submatrix W_{ii} of W is of full rank, the instruments can be solved in terms of the other variables and be eliminated from (11), leaving the first $n_X + n_x$ equations involving the Lagrange multipliers and the predetermined and forward-looking variables only.

Rewrite the $n_X + n_x + n_i$ FOCs as

$$\bar{A}' \begin{bmatrix} \xi_{t+1|t} \\ \Xi_t \end{bmatrix} = W \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} + \frac{1}{\delta}\bar{H}' \begin{bmatrix} \xi_t \\ \Xi_{t-1} \end{bmatrix}. \tag{14}$$

Combine them with the model equations (9) to get a system of $2(n_X + n_x) + n_i$ difference equations for $t \geq 0$,

$$\begin{bmatrix} \bar{H} & 0 \\ 0 & \bar{A}' \end{bmatrix} \begin{bmatrix} X_{t+1} \\ x_{t+1|t} \\ i_{t+1|t} \\ \hline \xi_{t+1|t} \\ \Xi_t \end{bmatrix} = \begin{bmatrix} \bar{A} & 0 \\ W & \frac{1}{\delta}\bar{H}' \end{bmatrix} \begin{bmatrix} X_t \\ x_t \\ i_t \\ \hline \xi_t \\ \Xi_{t-1} \end{bmatrix} + \begin{bmatrix} C \\ 0 \\ \hline 0 \\ 0 \end{bmatrix} \varepsilon_{t+1}. \tag{15}$$

X_t and Ξ_{t-1} are predetermined variables ($n_X + n_x$ in total)
x_t, i_t, and ξ_t are non-predetermined variables ($n_x + n_i + n_X$ in total).
Under suitable assumptions, this system has a unique solution for $t \geq 0$, given X_0 and $\Xi_{-1} = 0$. The solution uses the generalized Schur decomposition. Klein provides a detailed discussion of how this solution method relates to those of Blanchard and Kahn, Binder and Pesaran, King and Watson, Sims, and Uhlig.
The solution assumes the *saddlepoint property* emphasized by Blanchard and Kahn: The number of generalized eigenvalues with modulus larger than unity equals the number of non-predetermined variables, $n_x + n_i + n_X$.

Solving a system of linear difference equations with nonpredetermined variables
Consider the system

$$H \begin{bmatrix} y_{1,t+1} \\ \mathrm{E}_t y_{2,t+1} \end{bmatrix} = A \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_{t+1} \tag{16}$$

for $t \geq 0$.
y_{1t} is an n_1-vector of predetermined variables, y_{10} is given
y_{2t} is an n_2-vector of nonpredetermined variables,
ε_{t+1} is an iid random n_ε-vector with zero mean and covariance matrix I_{n_ε}.
The real matrices A and H are $n \times n$, where $n \equiv n_1 + n_2$, and the real matrix C is $n_1 \times n_\varepsilon$.
Take expectations conditional on information in period t and write the system as

$$H \begin{bmatrix} \mathrm{E}_t y_{1,t+1} \\ \mathrm{E}_t y_{2,t+1} \end{bmatrix} = A \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} \tag{17}$$

Following Klein, Sims, and Söderlind, use the generalized Schur decomposition (Golub and van Loan) of A and H:
There exists square possibly complex matrices Q, S, T, and Z such that

$$A = Q'TZ', \tag{18}$$

$$H = Q'SZ', \tag{19}$$

where Q' for a complex matrix denotes the complex conjugate transpose of Q (the transpose of the complex conjugate of Q).
(If $Q = [q_{jk}]$ has elements $q_{jk} = \operatorname{Re} q_{jk} + i \operatorname{Im} q_{jk}$, the complex conjugate of Q is the matrix $\bar{Q} = [\bar{q}_{jk}]$ with elements $\bar{q}_{jk} = \operatorname{Re} q_{jk} - i \operatorname{Im} q_{jk}$.)
The matrices Q and Z are unitary ($Q'Q = Z'Z = I$)
S and T are upper triangular
Sort the decomposition according to ascending modulus of the generalized eigenvalues, so $|\lambda_j| \geq |\lambda_k|$ for $j \geq k$. (Done by two programs by Sims, Qzdiv and Qzswitch.)
The generalized eigenvalues are the ratios of the diagonal elements of T and S, $\lambda_j = t_{jj}/s_{jj}$ ($j = 1, ..., n$). A generalized eigenvalue is infinity if $t_{jj} \neq 0$ and $s_{jj} = 0$ and zero if $t_{jj} = 0$ and $s_{jj} \neq 0$.

Assume the *saddlepoint property* (Blanchard and Kahn): The number of generalized eigenvalues with modulus larger than unity (the unstable eigenvalues) equals the number of non-predetermined variables. Thus, assume $|\lambda_j| > 1$ for $n_1 + 1 \leq j \leq n_1 + n_2$ and $|\lambda_j| < 1$ for $1 \leq j \leq n_1$. (For an exogenous predetermined variable with a unit root, we may allow $|\lambda_j| = 1$ for some $1 \leq j \leq n_1$. Also $\lambda_j = 1$ for any unity component of X_t — or subtract constant means.)
Define

$$\begin{bmatrix} \tilde{y}_{1t} \\ \tilde{y}_{2t} \end{bmatrix} \equiv Z' \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix}. \tag{20}$$

Interpret $\tilde{y}_{1t}$ as a complex vector of n_1 transformed predetermined variables and $\tilde{y}_{2t}$ as a complex vector of n_2 transformed non-predetermined variables. Premultiply the system (17) by Q and use (18)-(20) to write it as

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} \mathrm{E}_t \tilde{y}_{1,t+1} \\ \mathrm{E}_t \tilde{y}_{2,t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} \tilde{y}_{1t} \\ \tilde{y}_{2t} \end{bmatrix}, \tag{21}$$

where S and T have been partitioned conformably with $\tilde{y}_{1t}$ and $\tilde{y}_{2t}$.

Consider the lower block of (21),

$$S_{22}\,\mathrm{E}_t\tilde{y}_{2,t+1} = T_{22}\,\tilde{y}_{2t}. \tag{22}$$

Since the diagonal terms of S_{22} and T_{22} (s_{jj} and t_{jj} for $n_1+1 \le j \le n_1+n_2$) satisfy $|t_{jj}/s_{jj}| > 1$, the diagonal terms of T_{22} are nonzero, the determinant of T_{22} is nonzero, and T_{22} is invertible. Note that S_{22} may not be invertible. Then solve for $\tilde{y}_{2t}$ as

$$\tilde{y}_{2t} = J\mathrm{E}_t\tilde{y}_{2,t+1} = 0, \tag{23}$$

where the complex matrix J is given by

$$J \equiv T_{22}^{-1}S_{22}. \tag{24}$$

Exploit that the modulus of the diagonal terms of $T_{22}^{-1}S_{22}$ is less than one. Assume that $\mathrm{E}_t\tilde{y}_{2,t+\tau}$ is sufficiently bounded. Then $J^{\tau}\mathrm{E}_t\tilde{y}_{2,t+\tau} \to 0$ when $\tau \to \infty$. Note that J may not be invertible, since S_{22} may not be invertible.

By (20),

$$\begin{aligned} y_{1t} &= Z_{11}\tilde{y}_{1t}, & (25)\\ y_{2t} &= Z_{21}\tilde{y}_{1t}, & (26) \end{aligned}$$

where

$$Z \equiv \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \tag{27}$$

is partitioned conformably with y_{1t} and y_{2t}.

Under the assumption of the saddlepoint property, Z_{11} is square. Furthermore, assume that Z_{11} is *invertible*.

Solve for $\tilde{y}_{1t}$ in (25),

$$\tilde{y}_{1t} = Z_{11}^{-1}y_{1t}, \tag{28}$$

and use this in (26) to get

$$y_{2t} = Fy_{1t}, \tag{29}$$

where the real $n_2 \times n_1$ matrix F is given by

$$F \equiv Z_{21}Z_{11}^{-1}. \tag{30}$$

It remains to find a solution for $y_{1,t+1}$. By (23), the upper block of (21) is

$$S_{11}\mathrm{E}_t\tilde{y}_{1,t+1} = T_{11}\tilde{y}_{1t}.$$

Since the diagonal terms of S_{11} and T_{11} satisfy $|t_{jj}/s_{jj}| < 1$, all diagonal terms of S_{11} must be nonzero, so the determinant of S_{11} is nonzero, and S_{11} is invertible. Then solve for $\mathrm{E}_t\tilde{y}_{1,t+1}$ as

$$\mathrm{E}_t\tilde{y}_{1,t+1} = S_{11}^{-1}T_{11}\tilde{y}_{1t}.$$

By (25),

$$\begin{aligned}\mathrm{E}_t y_{1,t+1} &= Z_{11}\mathrm{E}_t\tilde{y}_{1,t+1} \\ &= Z_{11}S_{11}^{-1}T_{11}\tilde{y}_{1t} \\ &= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}y_{1t} \qquad (31)\end{aligned}$$

where (28) is used.

It follows that we can write the solution as

$$y_{1,t+1} = My_{1t} + C\varepsilon_{t+1}, \qquad (32)$$

where the real matrix M is given by

$$M \equiv Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1} \qquad (33)$$

Thus, the solution to the system (16) is given by (29) and (32) for $t \geq 0$.

(See Sims (2000) and Svensson (2005, app. B) for the case when ε_t is an arbitrary stochastic process.)

The solution to (15)

Let

$$\tilde{X}_t \equiv \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix}, \qquad \tilde{C} \equiv \begin{bmatrix} C \\ 0 \end{bmatrix}.$$

Then the solution can be written

$$\begin{aligned} x_t &= F_x\tilde{X}_t \qquad (34) \\ i_t &= F_i\tilde{X}_t, \qquad (35) \\ \tilde{X}_{t+1} &= M\tilde{X}_t + \tilde{C}\varepsilon_{t+1}, \qquad (36)\end{aligned}$$

F_x, F_i, and M depend on A, B, H, D, Λ, and δ, but are independent of C.

- Certainty equivalence of the commitment solution: It is independent of the covariance matrix of the shocks to X_t, CC', and the same as when that covariance matrix is zero.

Solution for the forward-looking Lagrange multiplier ξ_t,

$$\xi_t = F_\xi\tilde{X}_t,$$

not needed here.

The matrix F_i can be called the *optimal policy function* or the *optimal reaction function*.

The submatrices of the matrix M, F_x, and F_i,

$$M \equiv \begin{bmatrix} M_{XX} & M_{X\Xi} \\ M_{\Xi X} & M_{\Xi\Xi} \end{bmatrix}, \qquad F_x \equiv \begin{bmatrix} F_{xX} & F_{x\Xi} \end{bmatrix}, \qquad F_i \equiv \begin{bmatrix} F_{iX} & F_{i\Xi} \end{bmatrix},$$

are related according to

$$\begin{aligned} M_{XX} &\equiv A_{11} + A_{12}F_{xX} + B_1 F_{iX}, \\ M_{X\Xi} &\equiv A_{12}F_{x\Xi} + B_1 F_{i\Xi}. \end{aligned}$$

Note that, as is the case for non-predetermined variables, the one-period-ahead forecast errors of i_t and x_t are endogenous,

$$\begin{aligned} x_{t+1} - \mathrm{E}_t x_{t+1} &= F_x \tilde{C}\varepsilon_{t+1}, \\ i_{t+1} - \mathrm{E}_t i_{t+1} &= F_i \tilde{C}\varepsilon_{t+1}, \end{aligned}$$

since F_x and F_i are endogenous.
In a commitment equilibrium,

$$Y_t = D \begin{bmatrix} I & 0 \\ & F_x \\ & F_i \end{bmatrix} \tilde{X}_t \equiv \tilde{D}\tilde{X}_t,$$

$$L_t = \frac{1}{2} Y_t{}' \Lambda Y_t \equiv \frac{1}{2}\tilde{X}_t' \bar{W} \tilde{X}_t,$$

where $\tilde{D}$ is an $n_Y \times (n_X + n_x)$ matrix and $\bar{W} \equiv \tilde{D}'\Lambda\tilde{D}$ is an $(n_X + n_x) \times (n_X + n_x)$ matrix.

The equilibrium loss in any period $t \geq 0$ satisfies

$$\mathrm{E}_t \sum_{\tau=0}^{\infty} (1-\delta)\delta^\tau L_{t+\tau} = \frac{1}{2}[(1-\delta)\tilde{X}_t' V \tilde{X}_t + \delta w], \tag{37}$$

where V is an $(n_X + n_x) \times (n_X + n_x)$ matrix and w a scalar. The equilibrium loss satisfies the Bellman equation,

$$(1-\delta)\tilde{X}_t' V \tilde{X}_t + \delta w = (1-\delta)\tilde{X}_t' \bar{W} \tilde{X}_t + \delta \mathrm{E}_t[(1-\delta)\tilde{X}_{t+1}' V \tilde{X}_{t+1} + \delta w].$$

From this and (36) follows that V satisfies the Lyapunov equation,

$$V = \bar{W} + \delta M' V M, \tag{38}$$

and w satisfies

$$w = \mathrm{tr}\left\{ V \tilde{C}\tilde{C}' \right\}. \tag{39}$$

Furthermore, from (37) and (39) it follows that

$$\lim_{\delta \to 1-} \mathrm{E}_t \sum_{\tau=0}^{\infty} (1-\delta)\delta^\tau L_{t+\tau} = \frac{1}{2} w = \frac{1}{2}\mathrm{tr}\left\{ V \tilde{C}\tilde{C}' \right\}.$$

Commitment in a timeless perspective

Suppose the commitment is not made in period 0 but far into the past, or alternatively, that any commitment in any period t is restricted as if it had been made far into the past. This kind of commitment has been called a "commitment in a timeless perspective" by Woodford, cf. Svensson-Woodford. Then the condition (13) no longer applies, and the first-order condition (11) and the solution (34)-(36) holds for all $t = ... - 1, 0, 1, ...$ As noted by Svensson-Woodford, a simple way of finding the solution for commitment in a timeless perspective is to add the term $\frac{1-\delta}{\delta}\Xi'_{t-1}Hx_t$ to the commitment problem in period t, where Ξ_{t-1} is the Lagrange multiplier of the equations for the forward-looking variables from the optimization in period $t-1$. Then, the optimization problem in period t has the intertemporal loss function,

$$\mathrm{E}_t \sum_{\tau=0}^{\infty}(1-\delta)\delta^\tau L_{t+\tau} + \frac{1-\delta}{\delta}\Xi'_{t-1}Hx_t.$$

When this term is added, optimization under discretion in each period also results in the solution for commitment in a timeless perspective. This term is also related to the recursive saddlepoint method of Marcet and Marimon.

The Linear Quadratic Regulator (LQR) problem

Consider the problem (Anderson-Hansen-McGrattan-Sargent, Ljunqvist-Sargent):

$$V(X_t) \equiv \min_{\{i_{t+\tau}\}} \mathrm{E}_t \sum\nolimits_{\tau=0}^{\infty}(1-\delta)\delta^\tau L_{t+\tau},$$

subject to

$$L_t = \frac{1}{2}\begin{bmatrix} X_t \\ i_t \end{bmatrix}' W \begin{bmatrix} X_t \\ i_t \end{bmatrix},$$

$$X_{t+1} = AX_t + Bi_t + C\varepsilon_{t+1}, \tag{40}$$

where X_t is given.

The value function will be quadratic (linear model, quadratic loss function),

$$V(X_t) \equiv \frac{1}{2}[(1-\delta)X_t'VX_t + \delta w].$$

The Bellman equation can be written,

$$\frac{1}{2}[(1-\delta)X_t'VX_t+\delta w] = (1-\delta)\min_{i_t}\left\{\frac{1}{2}\begin{bmatrix} X_t \\ i_t \end{bmatrix}' W \begin{bmatrix} X_t \\ i_t \end{bmatrix} + \delta\mathrm{E}_t\frac{1}{2}[X_{t+1}'VX_{t+1} + \frac{\delta}{1-\delta}w]\right\} \tag{41}$$

subject to (40).

The first-order condition with respect to i_t is

$$Ji_t + KX_t = 0,$$

where the matrices J and K are defined as

$$\begin{aligned} J &\equiv R + \delta B'VB, \\ K &\equiv N' + \delta B'VA, \end{aligned}$$

where

$$W \equiv \begin{bmatrix} Q & N \\ N' & R \end{bmatrix},$$

is partitioned conformably with X_t and i_t. It follows that the solution for i_t can be written

$$i_t = FX_t, \tag{42}$$

where

$$F \equiv -J^{-1}K. \tag{43}$$

Using (42) and (43) in (41) results in the Riccati equation,

$$V = Q + \delta A'VA - K'J^{-1}K.$$

Thus, the solution F can be found by first solving the Riccati equation for V and then using (43).

In equilibrium, we have

$$X_{t+1} = MX_t + C\varepsilon_t,$$

where

$$M = A + BF.$$

Substitution into (41) gives

$$w = (1-\delta)\mathrm{E}_t[\varepsilon'_{t+1}C'VC\varepsilon_{t+1}] + \delta w,$$

$$w = \mathrm{E}_t[\varepsilon'_{t+1}C'VC\varepsilon_{t+1}] = \mathrm{E}_t\mathrm{tr}[\varepsilon'_{t+1}C'VC\varepsilon_{t+1}] = \mathrm{E}_t\mathrm{tr}[C'VC\varepsilon_{t+1}\varepsilon'_{t+1}] = \mathrm{tr}[C'VC],$$

$$w = \mathrm{tr}[VCC'].$$

(We have used $\mathrm{tr}\,[ABC] = \mathrm{tr}[BCA] = \mathrm{tr}[CAB]$.)

The LQR problem can also be solved with the Lagrange method: Set up the Lagrangian; find the FOCs, combine with (40) to get the system of difference equtions in X_t, i_t, and ξ_t; solve for i_t and ξ_t as functions of X_t.

Solving the Lyapunov equation for the matrix V

$$V = W + \delta M'VM,$$

where W and M are given $n \times n$ matrices. This equation can be solved using the relations

$$\mathrm{vec}(A + B) = \mathrm{vec}(A) + \mathrm{vec}(B)$$

$$\mathrm{vec}(ABC) = (C' \otimes A)\ \mathrm{vec}(B),$$

(where $\mathrm{vec}(A)$ denotes the vector of stacked column vectors of the matrix A, and $\otimes$ denotes the Kronecker product). This gives

$$\begin{aligned} \mathrm{vec}\,(V) &= \mathrm{vec}(W) + \delta \mathrm{vec}\,(M'VM) \\ &= \mathrm{vec}(W) + \delta\,(M' \otimes M')\,\mathrm{vec}\,(V)\,. \end{aligned}$$

Solving for $\mathrm{vec}\,(V)$ gives

$$\mathrm{vec}\,(V) = [I - \delta\,(M' \otimes M')]^{-1}\,\mathrm{vec}(\bar{W}),$$

where the matrix $I - \delta\,(M' \otimes M')$ must be invertible.
Also, note that

$$\begin{aligned} \mathrm{trace}(ABC) &= \mathrm{trace}(BCA) = \mathrm{trace}(CAB), \\ \mathrm{trace}(A + B) &= \mathrm{trace}(A) + \mathrm{trace}(B). \end{aligned}$$

The recursive saddlepoint (RSP) method of Marcet and Marimon

The problem to minimize (8) subject to (1) and (6)-(8) is not recursive. The forward-looking variables, x_t, depend on expected future forward-looking variables, (3). The RSP method of Marcet and Marimon provides a simple way to reformulate the problem as a dual recursive saddlepoint problem, so dynamic programming can be applied. The dual problem is then, except for being a saddlepoint problem, isomorphic to the standard backward-looking LQR problem.

Rewrite the Lagrangian as

$$\begin{aligned} \mathcal{L}_0 \;=\; & \mathrm{E}_0 \sum_{t=0}^{\infty} (1-\delta)\delta^t \begin{bmatrix} L_t + \Xi_t'(Hx_{t+1} - A_{21}X_t - A_{22}x_t - B_2 i_t) \\ + \xi_{t+1}'(X_{t+1} - A_{11}X_t - A_{12}x_t - B_1 i_t - C\varepsilon_{t+1}) \end{bmatrix} \\ & + \frac{1-\delta}{\delta}\xi_0'(X_0 - \bar{X}_0). \end{aligned}$$

The term Hx_{t+1}, dated $t+1$, appears in the first row of the Lagrangian. Therefore, not recursive.

However, because of (13), the discounted sum of the first row can be written

$$\begin{aligned} & \sum_{t=0}^{\infty} (1-\delta)\delta^t [L_t + \Xi_t'(Hx_{t+1} - A_{21}X_t - A_{22}x_t - B_2 i_t)] \;= \\ & \sum_{t=0}^{\infty} (1-\delta)\delta^t [L_t + \Xi_t'(-A_{21}X_t - A_{22}x_t - B_2 i_t) + \frac{1}{\delta}\Xi_{t-1}' H x_t]. \end{aligned} \tag{44}$$

Now all the terms within the bracket on the right side are dated t or earlier. The RSP method is in this case to let this term define the dual period loss.

$$
\begin{aligned}
\tilde{L}_t &\equiv L_t + \gamma_t'(-A_{21}X_t - A_{22}x_t - B_2 i_t) + \frac{1}{\delta}\Xi_{t-1}' H x_t \\
&\equiv L_t + L_t^1 \\
&\equiv \tilde{L}(X_t, \Xi_{t-1}; x_t, i_t, \gamma_t),
\end{aligned} \tag{45}
$$

Ξ_{t-1} is a new predetermined variable in period t, γ_t is introduced as a new control, Ξ_{t-1} and γ_t are related by the transition equation,

$$\Xi_t = \gamma_t. \tag{46}$$

The problem can then be reformulated as the recursive dual saddlepoint problem,

$$\max_{\{\gamma_t\}_{t\geq 0}} \min_{\{x_t, i_t\}_{t\geq 0}} \mathrm{E}_0 \sum_{t=0}^{\infty} (1-\delta)\delta^t \tilde{L}_t, \tag{47}$$

where the optimization is subject to (2), (46), and to X_0 and $\Xi_{-1} = 0$ given. The value function for the saddlepoint problem, starting in any period t, satisfies the Bellman equation

$$\tilde{V}(X_t, \Xi_{t-1}) \equiv \max_{\gamma_t} \min_{(x_t, i_t)} \{(1-\delta)\tilde{L}(X_t, \Xi_{t-1}; x_t, i_t, \gamma_t) + \delta \mathrm{E}_t \tilde{V}(X_{t+1}, \Xi_t)\},$$

subject to (2) and (46).

Define

$$\tilde{\imath}_t = \begin{bmatrix} x_t \\ i_t \\ \gamma_t \end{bmatrix},$$

and define $\tilde{W}$, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ such that

$$\tilde{L}_t = \frac{1}{2} \begin{bmatrix} \tilde{X}_t \\ \tilde{\imath}_t \end{bmatrix}' \tilde{W} \begin{bmatrix} \tilde{X}_t \\ \tilde{\imath}_t \end{bmatrix}, \tag{48}$$

$$\tilde{X}_{t+1} = \tilde{A}\tilde{X}_t + \tilde{B}\tilde{\imath}_t + \tilde{C}\varepsilon_{t+1}. \tag{49}$$

Then, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ satisfy

$$\tilde{A} \equiv \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{B} \equiv \begin{bmatrix} A_{12} & B_1 & 0 \\ 0 & 0 & I \end{bmatrix}, \qquad \tilde{C} \equiv \begin{bmatrix} C \\ 0 \end{bmatrix}$$

and $\tilde{W}$ satisfies

$$\tilde{W} = \begin{bmatrix} W_{XX} & 0 & W_{Xx} & W_{Xi} & -A_{21}' \\ 0 & 0 & \frac{1}{\delta}H & 0 & 0 \\ W_{Xx}' & \frac{1}{\delta}H' & W_{xx} & W_{xi} & -A_{22}' \\ W_{Xi}' & 0 & W_{xi}' & W_{ii} & -B_2' \\ -A_{21} & 0 & -A_{22} & -B_2 & 0 \end{bmatrix},$$

where W is partitioned conformably with X_t, x_t, and i_t according to

$$W \equiv \begin{bmatrix} W_{XX} & W_{Xx} & W_{Xi} \\ W'_{Xx} & W_{xx} & W_{xi} \\ W'_{Xi} & W'_{xi} & W_{ii} \end{bmatrix}.$$

The problem to minimize (47) subject to (49) and given $\tilde{X}_t$ is isomorphic to the LQR problem, except being a saddlepoint problem. However, the saddlepoint aspect does not affect the first-order conditions. It is easy to show that the first-order conditions of the saddlepoint problem are identical to those of the original problem.

Hence, use the standard solution for the LQR problem:
Value function

$$\tilde{V}(\tilde{X}_t) \equiv \frac{1}{2}[(1-\delta)\tilde{X}_t'\tilde{V}\tilde{X}_t + \delta\tilde{w}],$$

Bellman equation

$$(1-\delta)\tilde{X}_t'\tilde{V}\tilde{X}_t + \delta\tilde{w} = (1-\delta)\max_{\gamma_t}\min_{(x_t,i_t)}\{\begin{bmatrix} \tilde{X}_t \\ \tilde{\imath}_t \end{bmatrix}'\tilde{W}\begin{bmatrix} \tilde{X}_t \\ \tilde{\imath}_t \end{bmatrix} + \delta(\mathrm{E}_t\tilde{X}_{t+1}'\tilde{V}\tilde{X}_{t+1} + \frac{\delta}{1-\delta}\tilde{w})\} \tag{50}$$

subject to (49).
FOC with respect to $\tilde{\imath}_t$ is

$$J\tilde{\imath}_t + K\tilde{X}_t = 0,$$

where

$$\begin{aligned} J &\equiv R + \delta\tilde{B}'\tilde{V}\tilde{B}, \\ K &\equiv N' + \delta\tilde{B}'\tilde{V}\tilde{A}, \end{aligned}$$

$$\tilde{W} \equiv \begin{bmatrix} Q & N \\ N' & R \end{bmatrix},$$

$$Q \equiv \begin{bmatrix} W_{XX} & 0 \\ 0 & 0 \end{bmatrix}, \qquad N \equiv \begin{bmatrix} W_{Xx} & W_{Xi} & -A'_{21} \\ \frac{1}{\delta}H & 0 & 0 \end{bmatrix}, \qquad R \equiv \begin{bmatrix} W_{xx} & W_{xi} & -A'_{22} \\ W'_{xi} & W_{ii} & -B'_2 \\ -A_{22} & -B_2 & 0 \end{bmatrix}.$$

Solution for $\tilde{\imath}_t$

$$\tilde{\imath}_t = F\tilde{X}_t, \tag{51}$$

where

$$F \equiv -J^{-1}K. \tag{52}$$

Using (51) and (52) in (50) results in the Riccati equation,

$$\tilde{V} = Q + \delta\tilde{A}'\tilde{V}\tilde{A} - K'J^{-1}K.$$

Thus, the solution F can be found by first solving the Riccati equation for $\tilde{V}$ and then using (52).

The matrix F provides the solution not only to the saddlepoint problem but also to the original problem. The equilibrium dynamics will then be given by

$$\begin{aligned} \tilde{X}_{t+1} &= M\tilde{X}_t + \tilde{C}\varepsilon_{t+1}, \\ x_t &= F_x\tilde{X}_t, \\ i_t &= F_i\tilde{X}_t, \\ L_t &= \frac{1}{2}\tilde{X}'_t\bar{W}\tilde{X}_t, \end{aligned} \tag{53}$$

where

$$M \equiv \tilde{A} + \tilde{B}F,$$

the matrix F is partitioned conformably with x_t, i_t, and γ_t,

$$F \equiv \begin{bmatrix} F_x \\ F_i \\ F_\gamma \end{bmatrix},$$

and

$$\bar{W} \equiv \begin{bmatrix} I \;\; 0 \\ F_x \\ F_i \end{bmatrix}' W \begin{bmatrix} I \;\; 0 \\ F_x \\ F_i \end{bmatrix}.$$

F is also the solution to the original problem

Value function of the saddlepoint problem? $\tilde{V}$ and $\tilde{w}$ does not directly provide the value function of the original problem, because $\tilde{L}_t$ differs from L_t. Indeed, the matrix $\tilde{V}$ is not positive semidefinite.

Decompose the value function of the saddlepoint problem according to

$$\frac{1}{2}[(1-\delta)\tilde{X}_t'\tilde{V}\tilde{X}_t + \delta\tilde{w})] \equiv \frac{1}{2}[(1-\delta)\tilde{X}_t'V\tilde{X}_t + \delta w] + \frac{1}{2}[(1-\delta)\tilde{X}_t'V^1\tilde{X}_t + \delta w^1],$$

where

$$\frac{1}{2}[(1-\delta)\tilde{X}_t'V\tilde{X}_t + \delta w] \equiv \mathrm{E}_t \sum_{\tau=0}^{\infty}(1-\delta)\delta^{\tau-t}\frac{1}{2}\tilde{X}_{t+\tau}'\bar{W}\tilde{X}_{t+\tau} = \mathrm{E}_t \sum_{\tau=0}^{\infty}(1-\delta)\delta^{\tau-t}L_{t+\tau},$$

is the value function for the original problem (starting in period t with $\tilde{X}_t$ given).

Bellman equation for the original problem

$$\frac{1}{2}[(1-\delta)\tilde{X}_t'V\tilde{X}_t + \delta w] \equiv \frac{1}{2}(1-\delta)\tilde{X}_t'\bar{W}\tilde{X}_t + \frac{1}{2}\delta\mathrm{E}_t[(1-\delta)\tilde{X}_{t+1}'V\tilde{X}_{t+1} + \delta w],$$

V satisfies the Lyapunov equation,

$$V = \bar{W} + \delta M'VM,$$

and is positive semidefinite. The scalar w satisifies

$$w = \mathrm{tr}(V\tilde{C}\tilde{C}').$$

However, more direct way:

Note that, by (3), (45), and (46), identity

$$\frac{1}{2}[(1-\delta)\tilde{X}_t'V\tilde{X}_t + \delta w] \equiv \frac{1}{2}[(1-\delta)\tilde{X}_t'\tilde{V}\tilde{X}_t + \delta\tilde{w}] - (1-\delta)\frac{1}{\delta}\Xi_{t-1}'HF_x\tilde{X}_t,$$

$$\frac{1}{2}[(1-\delta)\tilde{X}_t'V^1\tilde{X}_t + \delta w^1] \equiv -(1-\delta)\frac{1}{\delta}\Xi_{t-1}'HF_x\tilde{X}_t \equiv -\frac{1}{2}(1-\delta)\frac{1}{\delta}(\Xi_{t-1}'HF_x\tilde{X}_t + \tilde{X}_t'F_x'H'\Xi_{t-1}).$$

Identification of terms implies $w^1 \equiv 0$. w and V are determined by

$$w = \tilde{w},$$

$$V = \tilde{V} - \begin{bmatrix} 0 & \frac{1}{\delta}F_{xX}'H' \\ \frac{1}{\delta}HF_{xX} & \frac{1}{\delta}(HF_{x\Xi} + F_{x\Xi}'H') \end{bmatrix} \equiv \begin{bmatrix} \tilde{V}_{XX} & \tilde{V}_{X\Xi} - \frac{1}{\delta}F_{xX}'H' \\ \tilde{V}_{\Xi X} - \frac{1}{\delta}HF_{xX} & \tilde{V}_{\Xi\Xi} - \frac{1}{\delta}(HF_{x\Xi} + F_{x\Xi}'H') \end{bmatrix},$$

$$\tilde{V} \equiv \begin{bmatrix} \tilde{V}_{XX} & \tilde{V}_{X\Xi} \\ \tilde{V}_{\Xi X} & \tilde{V}_{\Xi\Xi} \end{bmatrix}, \qquad F_x \equiv \begin{bmatrix} F_{xX} & F_{x\Xi} \end{bmatrix}.$$

In summary,

- The original problem is reformulated by incorporating the block of equations for the forward-looking variables, (3), in such a way that the resulting saddlepoint problem becomes recursive and isomorphic to the LQR problem.
- The solution to LQR problem is the solution to the original problem
- The value function for the original problem is identified.

Marcet-Marimon and Svensson (2006, appendix) shows that the recursive saddlepoint method can also be applied to problems that are not linear-quadratic.

From (44) and (3), it follows that

$$\mathrm{E}_t \sum_{\tau=0}^{\infty} (1-\delta)\delta^{\tau} \tilde{L}_{t+\tau} = \mathrm{E}_t \sum_{\tau=0}^{\infty} (1-\delta)\delta^{\tau} L_{t+\tau} + \frac{1-\delta}{\delta}\Xi'_{t-1} H x_t, \tag{54}$$

The intertemporal loss function for the dual problem and the original problem differ by

$$\frac{1-\delta}{\delta}\Xi'_{t-1} H x_t. \tag{55}$$

- Minimizing the right side of (54) under discretion will result in the optimal policy under commitment in a timeless perspective.
- Svensson-Woodford: A "commitment to continuity and predictability," a CB optimizing under discretion but taking into account previous expectations and plans in the form of adding (55) to its intertemporal loss function. That is, such a commitment means that the central bank applies the appropriate shadow price vector $\frac{1-\delta}{\delta}\Xi'_{t-1}$ from the previous period's optimization to the linear combination Hx_t of the current period's forward-looking variables. Such a commitment to a modified loss function then implies that optimization under discretion results in the optimal policy under commitment in a timeless perspective.

Using the recursive saddlepoint method to solve linear difference equations with forward-looking variables

Consider the system

$$
\begin{aligned}
X_{t+1} &= A_{11}X_t + A_{12}x_t + C\varepsilon_{t+1}, && (56)\\
\mathrm{E}_t H x_{t+1} &= A_{21}X_t + A_{22}x_t, && (57)
\end{aligned}
$$

and assume that it has a unique solution

$$x_t = F_x X_t. \tag{58}$$

This solution can be found with the generalized Schur decomposition, as in Klein and demonstrated above.

The solution can also be found with the recursive saddlepoint problem. Let

$$L(X_t, x_t) \equiv \frac{1}{2}\begin{bmatrix} X_t \\ x_t \end{bmatrix}' W \begin{bmatrix} X_t \\ x_t \end{bmatrix},$$

where W is any positive semidefinite matrix, and let

$$\tilde{L}(\tilde{X}_t; \tilde{\imath}_t) \equiv L(X_t, x_t) + \gamma_t'(-A_{21}X_t - A_{22}x_t) + \frac{1}{\delta}\Xi_{t-1}' H x_t \equiv \frac{1}{2}\begin{bmatrix} \tilde{X}_t \\ \tilde{\imath}_t \end{bmatrix}' \tilde{W} \begin{bmatrix} \tilde{X}_t \\ \tilde{\imath}_t \end{bmatrix},$$

$$\tilde{X}_{t+1} = \tilde{A}\tilde{X}_t + \tilde{B}\tilde{\imath}_t + \tilde{C}\varepsilon_t,$$

where

$$\tilde{X}_t \equiv \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix}, \qquad \tilde{\imath}_t \equiv \begin{bmatrix} x_t \\ \gamma_t \end{bmatrix},$$

$$\tilde{W} = \begin{bmatrix} W_{XX} & 0 & W_{Xx} & -A_{21}' \\ 0 & 0 & \frac{1}{\delta}H & 0 \\ W_{Xx}' & \frac{1}{\delta}H' & W_{xx} & -A_{22}' \\ -A_{21} & 0 & -A_{22} & 0 \end{bmatrix},$$

where W is partitioned conformably with X_t and x_t according to

$$W \equiv \begin{bmatrix} W_{XX} & W_{Xx} \\ W_{Xx}' & W_{xx} \end{bmatrix}.$$

Then we can apply the recursive saddlepoint method as above. This will result in the solution

$$\tilde{\imath}_t \equiv \begin{bmatrix} x_t \\ \gamma_t \end{bmatrix} = \tilde{F}\tilde{X}_t \equiv \begin{bmatrix} \tilde{F}_{xX} & \tilde{F}_{x\Xi} \\ \tilde{F}_{\gamma X} & \tilde{F}_{\gamma\Xi} \end{bmatrix} \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix},$$

where

$$\tilde{F}_{xX} \equiv F_x, \qquad \tilde{F}_{x\Xi} \equiv 0. \tag{59}$$

Here (59) should be demonstrated in detail, but we realize that it must be true when there is a unique (nonbubble) solution (58).

Note that, since there are no degrees of freedom for x_t, the solution (59) for x_t does not depend on the matrix W. The solution for γ_t will depend on W.

Optimization under discretion: The discretion equilibrium

Let (1) be the model equations. Let the period loss function be

$$L_t = \frac{1}{2}\begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}' W \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}, \tag{60}$$

and let the intertemporal loss function in period t be

$$\mathrm{E}_t \sum_{\tau=0}^{\infty}(1-\delta)\delta^{\tau} L_{t+\tau},$$

where $0 < \delta < 1$.

Consider the decision problem to choose i_t in period t to minimize the intertemporal loss function under discretion, that is, subject to (1), X_t given, and

$$i_{t+1} = F_{t+1}X_{t+1} \tag{61}$$
$$x_{t+1} = G_{t+1}X_{t+1}, \tag{62}$$

where F_{t+1} and G_{t+1} are determined by the decision problem in period $t+1$. Both F_{t+1} and G_{t+1} are assumed known in period t; only G_{t+1} will matter for the decision problem in period t.

Oudiz and Sachs derive an algorithm for the solution of this problem (with $H = I$), which is further discussed in Backus and Driffill , Currie and Levin, and Söderlind.

First, take the expectation in period t of (1),

$$\begin{bmatrix} X_{t+1|t} \\ Hx_{t+1|t} \end{bmatrix} = A \begin{bmatrix} X_t \\ x_t \end{bmatrix} + Bi_t. \tag{63}$$

Second, using (62) and the upper block of (63) results in

$$x_{t+1|t} = G_{t+1}X_{t+1|t} = G_{t+1}(A_{11}X_t + A_{12}x_t + B_1 i_t). \tag{64}$$

The lower block of (1) is

$$Hx_{t+1|t} = A_{21}X_t + A_{22}x_t + B_2 i_t. \tag{65}$$

Multiplying (64) by H, setting the result equal to (65), and solving for x_t gives

$$x_t = \bar{A}_t X_t + \bar{B}_t i_t, \tag{66}$$

where

$$\bar{A}_t \equiv (A_{22} - HG_{t+1}A_{12})^{-1}(HG_{t+1}A_{11} - A_{21}), \tag{67}$$
$$\bar{B}_t \equiv (A_{22} - HG_{t+1}A_{12})^{-1}(HG_{t+1}B_1 - B_2) \tag{68}$$

(Assume that $A_{22} - HG_{t+1}A_{12}$ is invertible). Using (66) in the upper block of (1) then gives

$$X_{t+1} = \tilde{A}_t X_t + \tilde{B}_t i_t + C\varepsilon_{t+1}, \tag{69}$$

where

$$\tilde{A}_t \equiv A_{11} + A_{12}\bar{A}_t, \tag{70}$$
$$\tilde{B}_t \equiv B_1 + A_{12}\bar{B}_t. \tag{71}$$

Third, using (66) in (60) leads to

$$L_t = \frac{1}{2}\begin{bmatrix} X_t \\ i_t \end{bmatrix}' \begin{bmatrix} Q_t & N_t \\ N_t' & R_t \end{bmatrix} \begin{bmatrix} X_t \\ i_t \end{bmatrix}, \tag{72}$$

where

$$\begin{aligned} Q_t &\equiv W_{XX} + W_{Xx}\bar{A}_t + \bar{A}_t' W_{Xx}' + \bar{A}_t' W_{xx}\bar{A}_t, & (73)\\ N_t &\equiv W_{Xx}\bar{B}_t + \bar{A}_t' W_{xx}\bar{B}_t + W_{Xi} + \bar{A}_t' W_{xi}, & (74)\\ R_t &\equiv W_{ii} + \bar{B}_t' W_{xx}\bar{B}_t + \bar{B}_t' W_{xi} + W_{xi}'\bar{B}_t. & (75)\end{aligned}$$

Fourth, since the loss function is quadratic and the constraints are linear, it follows that the optimal value of the problem will be quadratic. In period $t+1$ the optimal value will depend on X_{t+1} and can hence be written $\frac{1}{2}[(1-\delta)X_{t+1}'V_{t+1}X_{t+1} + \delta w_{t+1}]$, where V_{t+1} is a positive semidefinite matrix and w_{t+1} is a scalar independent of X_{t+1}. Both V_{t+1} and w_{t+1} are assumed known in period t. Then the optimal value of the problem in period t is associated with the positive semidefinite matrix V_t and the scalar w_t, and satisfies the Bellman equation

$$\frac{1}{2}[(1-\delta)X_t'V_tX_t + \delta w_t] \equiv (1-\delta)\min_{i_t}\left\{L_t + \delta \mathrm{E}_t\frac{1}{2}[X_{t+1}'V_{t+1}X_{t+1} + \frac{\delta}{1-\delta}w_{t+1}]\right\}, \tag{76}$$

subject to (69) and (72).

Indeed, the problem has been transformed to a standard LQR problem without forward-looking variables, albeit in terms of X_t and with time-varying parameters. The first-order condition is, by (72) and (76),

$$\begin{aligned} 0 &= X_t'N_t + i_t'R_t + \delta \mathrm{E}_t[X_{t+1}'V_{t+1}\tilde{B}_t] \\ &= X_t'N_t + i_t'R_t + \delta(X_t'\tilde{A}_t' + i_t'\tilde{B}_t')V_{t+1}\tilde{B}_t. \end{aligned}$$

The first-order condition can be solved for the reaction function

$$i_t = F_tX_t, \tag{77}$$

where

$$F_t \equiv -\,(R_t + \delta\tilde{B}_t'V_{t+1}\tilde{B}_t)^{-1}(N_t' + \delta\tilde{B}_t'V_{t+1}\tilde{A}_t) \tag{78}$$

(Assume that $R_t + \delta\tilde{B}_t'V_{t+1}\tilde{B}_t$ is invertible). Using (77) in (66) gives

$$x_t = G_tX_t,$$

where

$$G_t \equiv \bar{A}_t + \bar{B}_tF_t. \tag{79}$$

Furthermore, using (77) in (76) and identifying terms result in

$$V_t \equiv Q_t + N_tF_t + F_t'N_t' + F_t'R_tF_t + \delta(\tilde{A}_t + \tilde{B}_tF_t)'V_{t+1}(\tilde{A}_t + \tilde{B}_tF_t). \tag{80}$$

Finally, the above equations ((67), (68), (70), (71), (73)–(75), (78), (79), and (80)) define a mapping from (G_{t+1}, V_{t+1}) to (G_t, V_t), which also determines F_t. The solution to the problem is a fixed point (G, V) of the mapping and a corresponding F. It can be obtained as the limit of (G_t, V_t) when $t \to -\infty$. The solution thus satisfies the corresponding steady-state matrix equations.

Thus, the instrument i_t and the forward-looking variables x_t will be linear functions,

$$i_t = FX_t, \tag{81}$$
$$x_t = GX_t, \tag{82}$$

where the corresponding F and G satisfy the corresponding steady-state equations. The matrix F can be called the equilibrium policy function or the equilibrium reaction function.

The resulting equation for X_t is

$$X_{t+1} = MX_t + C\varepsilon_{t+1}, \tag{83}$$

where

$$M \equiv \tilde{A} + \tilde{B}F$$

where $\tilde{A}$ and $\tilde{B}$ is the fixed point of the mapping from $(\tilde{A}_{t+1}, \tilde{B}_{t+1})$ to $(\tilde{A}_t, \tilde{B}_t)$.

It also follows that F, G, $\tilde{A}$, and $\tilde{B}$ depend on A, B, H, W, and δ, but are independent of C. This demonstrates the certainty equivalence of the discretionary equilibrium.

In a discretion equilibrium,

$$Y_t = D \begin{bmatrix} I \\ G \\ F \end{bmatrix} X_t \equiv \tilde{D}X_t,$$

$$L_t = \frac{1}{2}Y_t'\Lambda Y_t \equiv \frac{1}{2}X_t'\bar{W}X_t,$$

where $\tilde{D}$ is an $n_Y \times n_X$ matrix and $\bar{W} \equiv \frac{1}{2}\tilde{D}'\Lambda\tilde{D}$ is an $n_X \times n_X$ matrix.

The equilibrium loss in any period $t \geq 0$ will satisfy

$$\mathrm{E}_t \sum_{\tau=0}^{\infty}(1-\delta)\delta^\tau L_{t+\tau} = \frac{1}{2}[(1-\delta)X_t'VX_t + \delta w],$$

where the $n_X \times n_X$ matrix V and the scalar w, the fixed point of the mapping from (V_{t+1}, w_{t+1}) to (V_t, w_t), satisfy

$$V = \bar{W} + \delta M'VM,$$
$$w = \mathrm{tr}[VCC'].$$

The equilibrium loss obviously depends on C.

One might think that the discretion solution can also be found by combining (9) with the first-order condition (12) for $t \geq 0$. This solution is generally *not* correct. It amounts to treating expectations $x_{t+1|t}$ as exogenous. This is consistent with (62) only in the special case of all predetermined variables in the vector X_t being exogenous, in which case $x_{t+1|t} = GX_{t+1|t}$ is independent of i_t. However, if some predetermined variables are endogenous, $X_{t+1|t}$ and thereby $x_{t+1|t}$ will depend on i_t, which is taken into account in the Bellman equation derived above. The reason why the first-order conditions (12) for $t \geq 0$ give the correct discretion solution in the model of Svensson-Woodford is that all predetermined variables are exogenous there.

Relation to Sims's Gensys

Consider the system of equations (16) and introduce the vector of endogenous expectational errors of the nonpredetermined variables,

$$\eta_t \equiv y_{2t} - \mathrm{E}_{t-1} y_{2t},$$

which have the obvious property that $\mathrm{E}_{t-1}\eta_t = 0$. Using this to substitute for $\mathrm{E}_t y_{2,t+1}$ in (16), the latter can be written in the form

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi \varepsilon_t + \Pi \eta_t, \tag{84}$$

where

$$y_t \equiv \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix}, \qquad \Gamma_0 \equiv H, \qquad \Gamma_1 \equiv A, \qquad \Psi \equiv \begin{bmatrix} C \\ 0 \end{bmatrix}, \qquad \Pi \equiv H_2,$$

where $H \equiv [H_1 \;\; H_2]$ is partitioned conformably with y_{1t} and y_{2t}. Sims uses the generalized Schur decomposition to find a solution to (84) with a more elaborate method than the one used above, explicitly taking into account the condition $\mathrm{E}_{t-1}\eta_t = 0$ and the somewhat complex restrictions this implies.

Sims also deals with the case when ε_t is an arbitrary exogenous stochastic process and not necessarily a zero-mean iid shocks as above. Then the solution can be expressed as

$$y_t = \Theta_1 y_{t-1} + \Theta_0 \varepsilon_t + \Theta_y \sum_{\tau=0}^{\infty} \Theta_f^{\tau} \Theta_\theta \mathrm{E}_t \varepsilon_{t+1+\tau},$$

where Θ_0 and Θ_1 are real matrices, Θ_y, Θ_f, and Θ_θ are complex matrices, and $\Theta_y \Theta_f^{\tau} \Theta_\theta$ for any integer $\tau \geq 0$ is a real matrix. Svensson (2005, appendix) solves (16) with Klein's method when ε_t is an arbitrary exogenous stochastic process and expresses the solutions as

$$\begin{aligned} y_{2t} &= F y_{1t} + Z_t, \\ y_{1,t+1} &= M y_{1t} + N \mathrm{E}_t Z_{t+1} + PC\mathrm{E}_t \varepsilon_{t+1} + C(\varepsilon_{t+1} - \mathrm{E}_t \varepsilon_{t+1}), \\ Z_t &\equiv \sum_{\tau=0}^{\infty} R_\tau C \mathrm{E}_t \varepsilon_{t+1+\tau}, \end{aligned}$$

where F, M, N, P and $\{R_\tau\}_{\tau=0}^{\infty}$ are real matrices of appropriate dimension.
Klein provides a detailed discussion of how the solution method used in these notes relates to those of Blanchard and Kahn, Binder and Pesaran, King and Watson, Sims, and Uhlig.

How to go from (16),

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t|t-1} \end{bmatrix} = A \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_t$$

to (84)?
Simply introduce the endogenous forecast errors for the non-predetermined variables,

$$\eta_t \equiv y_{2t} - y_{2,t|t-1}.$$

Substitute for $y_{2t|t-1}$:

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = A \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} C \\ 0 \end{bmatrix} \varepsilon_t - \begin{bmatrix} H_{12} \\ H_{22} \end{bmatrix} \eta_t.$$

This system is the same form as

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi \varepsilon_t + \Pi \eta_t.$$

The recursive saddlepoint method for a nonlinear problem

Consider the nonlinear problem

$$\min_{\{i_t\}_{t\geq 0}} \mathrm{E}_0 \sum_{t=0}^{\infty}(1-\delta)\delta^t L_t, \tag{85}$$

where the period loss function is

$$L_t = L(X_t, x_t, i_t, s_t), \tag{86}$$

the constraints are

$$X_{t+1} = A_1(X_t, x_t, i_t, s_{t+1}), \tag{87}$$

$$\mathrm{E}_t H(X_{t+1}, x_{t+1}, s_{t+1}) = A_2(X_t, x_t, i_t, s_t), \tag{88}$$

where $\{s_t\}$ is an exogenous Markov process, $A_1(\cdot)$, $A_2(\cdot)$, and $H(\cdot)$ are vector-valued functions of the same dimension as X_t, x_t, and x_t, respectively, and where X_0 and s_0 are given. Equation (87) determines X_{t+1} given X_t, x_t, i_t, and s_{t+1}. The function $A_2(X_t, x_t, i_t, s_t)$ is assumed to be invertible with respect to x_t, so equation (88) determines x_t given X_t, i_t, s_t, and expectations $\mathrm{E}_t H(X_{t+1}, x_{t+1}, s_{t+1})$.

Write the Lagrangian as

$$\begin{aligned}\mathcal{L}_0 &= \mathrm{E}_0 \sum_{t=0}^{\infty}(1-\delta)\delta^t \left\{ \begin{array}{l} L_t + \Xi_t'[\mathrm{E}_t H(X_{t+1}, x_{t+1}, s_{t+1}) - A_2(X_t, x_t, i_t, s_t)] \\ + \xi_{t+1}'[X_{t+1} - A_1(X_t, x_t, i_t, s_{t+1})] \end{array} \right\} \\ &= \mathrm{E}_0 \sum_{t=0}^{\infty}(1-\delta)\delta^t \left\{ \begin{array}{l} L_t + \Xi_t'[H(X_{t+1}, x_{t+1}, s_{t+1}) - A_2(X_t, x_t, i_t, s_t)] \\ + \xi_{t+1}'[X_{t+1} - A_1(X_t, x_t, i_t, s_{t+1})] \end{array} \right\}, \end{aligned} \tag{89}$$

where Ξ_t is the vector of Lagrange multipliers for (88), ξ_{t+1} is the vector of Lagrange multipliers for (88), and the second equality follows from the law of iterated expectations. The problem is not recursive, since the term $H(X_{t+1}, x_{t+1}, s_{t+1})$, which depends on the forward-looking variable x_{t+1}, appears in the first line of of (89). However, note that the discounted sum of the first line of (89) can be written

$$\sum_{t=0}^{\infty}(1-\delta)\delta^t\{L_t + \Xi_t'[H(X_{t+1}, x_{t+1}, s_{t+1}) - A_2(X_t, x_t, i_t, s_t)]\} =$$

$$\sum_{t=0}^{\infty}(1-\delta)\delta^t\{L_t - \Xi_t' A_2(X_t, x_t, i_t, s_t) + \frac{1}{\delta}\Xi_{t-1}' H(X_t, x_t, s_t)\},$$

where $\Xi_{-1} = 0$. Now all the terms within the curly brackets on the right side are dated t or earlier. The recursive saddlepoint method is in this case to let this expression within the curly brackets define the dual period loss. More precisely, the dual period loss is defined as

$$\tilde{L}_t \equiv L_t - \gamma_t' A_2(X_t, x_t, i_t, s_t) + \frac{1}{\delta}\Xi_{t-1}' H(X_t, x_t, s_t) \equiv \tilde{L}(X_t, \Xi_{t-1}; x_t, i_t, \gamma_t; s_t),$$

where Ξ_{t-1} is a predetermined variable in period t and γ_t is an additional control variable, and where Ξ_{t-1} and γ_t are related by the dynamic equation,

$$\Xi_t = \gamma_t. \tag{90}$$

Marcet and Marimon show that the problem can then be reformulated as the recursive saddlepoint problem,

$$\max_{\{\gamma_t\}_{t\geq 0}} \min_{\{x_t, i_t\}_{t\geq 0}} \mathrm{E}_0 \sum_{t=0}^{\infty} (1-\delta)\delta^t \tilde{L}_t,$$

where the optimization is subject to (87), (90), and X_0 and Ξ_{-1} given. The value function for the saddlepoint problem, starting in any period t, satisfies the Bellman equation

$$\tilde{V}(\tilde{X}_t; s_t) \equiv \max_{\gamma_t} \min_{(x_t, i_t)} \{(1-\delta)\tilde{L}(\tilde{X}_t; \tilde{\imath}_t; s_t) + \delta \mathrm{E}_t \tilde{V}(\tilde{X}_{t+1}; s_{t+1})\},$$

subject to (87) and (90), where $\tilde{X}_t \equiv (X_t', \Xi_{t-1}')'$ and $\tilde{\imath}_t \equiv (x_t', i_t', \gamma_t')'$. The optimal policy function for the saddlepoint problem will be

$$\tilde{\imath}_t = F(\tilde{X}_t, s_t) \equiv \begin{bmatrix} F_x(\tilde{X}_t, s_t) \\ F_i(\tilde{X}_t, s_t) \\ F_\gamma(\tilde{X}_t, s_t) \end{bmatrix}.$$

It follows that the solution for the original problem is

$$\begin{aligned} x_t &= F_x(\tilde{X}_t, s_t), \\ i_t &= F_i(\tilde{X}_t, s_t), \\ L_t &= L[X_t, F_x(\tilde{X}_t, s_t), F_i(\tilde{X}_t, s_t), s_t] \;\equiv\; \bar{L}(\tilde{X}_t, s_t), \\ \tilde{X}_{t+1} &= \begin{bmatrix} A_1[X_t, F_x(\tilde{X}_t, s_t), F_i(\tilde{X}_t, s_t), s_{t+1}] \\ F_\gamma(\tilde{X}_t, s_t) \end{bmatrix} \;\equiv\; M(\tilde{X}_t, s_t, s_{t+1}). \end{aligned}$$

The value function for the original problem satisfies the Bellman equation

$$V(\tilde{X}_t; s_t) \equiv (1-\delta)\bar{L}(\tilde{X}_t, s_t) + \delta \mathrm{E}_t V[M(\tilde{X}_t, s_t, s_{t+1}), s_{t+1}].$$

This value function is related to the value function of the dual problem by

$$V(\tilde{X}_t; s_t) \equiv \tilde{V}(\tilde{X}_t; s_t) - \frac{1-\delta}{\delta}\Xi_{t-1}' H[X_t, F_x(\tilde{X}_t, s_t), s_t].$$

Expressing the constraint in the form (88) does not seem restrictive. Suppose, for instance, that the constraint (88) is replaced by a constraint in the form

$$\mathrm{E}_t \sum\nolimits_{\tau=1}^{\infty} \delta^\tau G(X_{t+\tau}, x_{t+\tau}, s_{t+\tau}) = A_2(X_t, x_t, i_t, s_t) \tag{91}$$

(this case is also treated in Marcet and Marimon). The constraint can easily be rewritten in the form (88). First, introduce the additional forward-looking variable

$$x_{2t} \equiv \mathrm{E}_t \sum\nolimits_{\tau=0}^{\infty} \delta^\tau G(X_{t+\tau}, x_{t+\tau}, s_{t+\tau}) = G(X_t, x_t, s_t) + \delta \mathrm{E}_t x_{2,t+1}.$$

Second, replace (91) by the two constraints

$$\begin{aligned}\delta \mathrm{E}_t x_{2,t+1} &= -G(X_t, x_t, s_t) + x_{2t},\\ 0 &= A_2(X_t, x_t, i_t, x_t) + G(X_t, x_t, s_t) - x_{2t}.\end{aligned}$$

These two constraints are obviously in the same form as (88), since they can be written

$$\mathrm{E}_t \tilde{H}(X_{t+1}, x_{t+1}, x_{2,t+1}, s_{t+1}) = \tilde{A}_2(X_t, x_t, x_{t2}, i_t, s_t),$$

where

$$\begin{aligned}\tilde{H}(X_t, x_t, x_{2t}, s_t) &\equiv \begin{bmatrix} \delta x_{2t} \\ 0 \end{bmatrix},\\ \tilde{A}_2(X_t, x_t, x_{t2}, i_t, s_t) &\equiv \begin{bmatrix} -G(X_t, x_t, s_t) + x_{2t} \\ A_2(X_t, x_t, i_t, x_t) + G(X_t, x_t, s_t) - x_{2t} \end{bmatrix}.\end{aligned}$$

Note that it should be possible to use the recursive saddlepoint method to solve nonlinear difference equations with forward-looking variables, as in the linear case.