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Monetary Policy with Model Uncertainty:

Distribution Forecast Targeting

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1 Introduction

- Optimal monetary policy under uncertainty
- Additive uncertainty, linear-quadratic (LQ) model:
  Certainty equivalence (CE), "mean forecast targeting"
- Broader forms of uncertainty:
  No CE, policy design complicated, "distribution forecast targeting"
- Onatski-Williams: Specify structure of model uncertainty!

- This paper:
  Markov jump-linear-quadratic (MJLQ) framework
  Relatively general form of model uncertainty, still tractable
  Model shifts between alternative "modes"
  - Simple i.i.d. random model deviations
  - Serially correlated model deviations
  - Estimable regime-switching models
  - More complex structural uncertainty:
    Very different models (backward- or forward-looking, number of leads and lags, etc.)
  - Both observable and unobservable modes
  - Time-varying central-bank (CB) judgment, for instance about the state of model uncertainty
  - Subjective and/or objective (transition) probability distributions over modes
  - (Endogenous transition probabilities)
  - Etc., etc.
  - Work in progress: Learning, Bayesian updating

- Related literature
  - Aoki 67, Chow 73: Multiplicative uncertainty in LQ model w/o forward-looking variables
  - Costa-Fragoso-Marques 05, others: MJLQ framework w/o forward-looking variables, observable modes, optimal policy
  - Marcet-Marimon 98: Recursive saddlepoint (RSP) method
  - Zampolli 05, Blake-Zampolli 05: similar, forward-looking variables, observable modes, less general

- Characteristics
  - General treatment of forward-looking variables w/ RSP method
  - Unobservable modes
  - Arbitrary time-varying/time-invariant policy functions
- Algorithms for solutions
- Compute and plot internally consistent distribution forecasts (fan charts) of target variables and instruments
- Extend CE and "mean forecast targeting" to non-CE and "distribution forecast targeting."

Outline

2. MJLQ model, extend to forward-looking model (recursive saddlepoint model, Marcet-Marimon), solution algorithm
3. Interpretation of model uncertainty
4. Two examples, empirical regime-switching models of US economy
   - 4.1 Backward-looking (Rudebusch-Svensson)
   - 4.2 Forward-looking (Lindé)

5 and 6. Arbitrary time-varying and time-invariant policy functions (instrument rules/paths, targeting rules)

7. Unobservable modes
8. Conclusions

Appendix. Technical details

2 The model

Economy w/ CB

Relatively general model uncertainty

Different CB info and judgment (consistent w/ Svensson, "Monetary policy w/ judgment")

2.1 Baseline model

$$X_{t+1} = A_{11,t+1}X_t + A_{12,t+1}x_t + B_{1,t+1}i_t + C_{t+1}\varepsilon_{t+1} \quad (1)$$

$$\mathrm{E}_t H_{t+1}x_{t+1} = A_{21,t}X_t + A_{22,t}x_t + B_{2,t}i_t, \quad (2)$$

X_t predetermined variables

x_t forward-looking variables

i_t CB instruments (control variables)

ε_t i.i.d. shock, $N(0, I)$

$A_{22,t}$ nonsingular, equation (2) determines x_t

(No restriction excluding ε_t from (2): Add predetermined variables)

E_t conditional expectation (information to be specified)

CB intertemporal loss function,

$$\mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^\tau L_{t+\tau}, \quad (3)$$

Y_t target variables

$$Y_t \equiv D_t \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}$$

L_t period loss

$$L_t \equiv Y_t' \Lambda_t Y_t \equiv \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}' W_t \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} \quad (4)$$

Λ_t and $W_t \equiv D_t' \Lambda_t D_t$ symmetric, positive semidefinite

Matrices $A_{11,t}$, $A_{12,t}$, $B_{1,t}$, C_t, H_t, $A_{21,t}$, $A_{22,t}$, $B_{2,t}$, W_t random, can take n different values in period t, corresponding to n *modes* $j_t = 1, 2, ..., n$.
$A_{11,t} = A_{11j_t}$, $A_{12,t} = A_{12j_t}$, etc., for $j_t = 1, 2, ..., n$.

- Modes j_t follow Markov process w/ constant transition probabilities
$$P_{jk} \equiv \Pr\{j_{t+1} = k \mid j_t = j\} \qquad (j, k = 1, ..., n). \tag{5}$$
ε_t and modes j_t independently distributed
(covariance of additive shocks $C_{j_t}C'_{j_t}$ depends on j_t).
P transition matrix $[P_{jk}]$
$p_{jt} \equiv \Pr\{j_t = j\}$ ($j = 1, ..., n$) probability of mode j_t in period t
$p_t \equiv (p_{1t}, ..., p_{nt})'$ probability distribution of modes in period t
$$p_{t+1} = P'p_t.$$
$\bar{p}$ stationary distribution
$$\bar{p} = P'\bar{p}.$$

- CB info beginning of period t:
X_t, j_t, ε_t, X_{t-1}, j_{t-1}, ε_{t-1}, x_{t-1}, i_{t-1}, ... ; matrices, distributions
Section 7: j_t unobserved, distribution p_t known
- Minimize (3) subject to (4), (1), (2), X_t given
(commitment, timeless perspective; Woodford, Svensson-Woodford)
- Control-theory literature (Costa-Fragoso-Marques)
MJLQ systems w/o forward-looking variables, recursive
w/ forward-looking variables, nonrecursive

2.2 The recursive saddlepoint method

Dual recursive saddlepoint problem (Marcet-Marimon)

$$\max_{\{\gamma_{t+\tau}\}_{\tau\geq 0}} \min_{\{x_{t+\tau},i_{t+\tau}\}_{\tau\geq 0}} \mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^\tau \tilde{L}_{t+\tau}, \tag{6}$$

Dual period loss function

$$\tilde{L}_{t+\tau} \equiv \begin{bmatrix} \tilde{X}_{t+\tau} \\ \tilde{\imath}_{t+\tau} \end{bmatrix}' \tilde{W}_{j_{t+\tau}} \begin{bmatrix} \tilde{X}_{t+\tau} \\ \tilde{\imath}_{t+\tau} \end{bmatrix}, \tag{7}$$

Dual model

$$\tilde{X}_{t+\tau+1} = \tilde{A}_{j_{t+\tau+1}} \tilde{X}_{t+\tau} + \tilde{B}_{j_{t+\tau+1}} \tilde{\imath}_{t+\tau} + \tilde{C}_{j_{t+\tau+1}} \varepsilon_{t+\tau+1} \qquad (\tau \geq 0), \tag{8}$$

$\tilde{X}_t$ and j_t given
$\tilde{X}_t$ dual predetermined variables, $\tilde{\imath}_t$ dual control variables

$$\tilde{X}_t \equiv \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix}, \qquad \tilde{\imath}_t = \begin{bmatrix} x_t \\ i_t \\ \gamma_t \end{bmatrix}. \tag{9}$$

Ξ_{t-1} Lagrange multipliers for (2) in period $t-1$
(history dependence; Woodford, Svensson-Woodford)
γ_t Lagrange multipliers for (2) in period t, control variables in period t.
Added dynamic equation (in (8)

$$\Xi_t = \gamma_t$$

Dual matrices $\tilde{W}_{j_t}$, $\tilde{A}_{j_{t+1}}$, $\tilde{B}_{j_{t+1}}$, $\tilde{C}_{j_{t+1}}$ satisfy

$$\begin{aligned} \tilde{L}_t &\equiv L_t + \gamma_t'(-A_{21j_t}X_t - A_{22j_t}x_t - B_{2j_t}i_t) + \frac{1}{\delta}\Xi_{t-1}' H_{j_t} x_t \\ &\equiv \tilde{X}_t' Q_{j_t} \tilde{X}_t + 2\tilde{X}_t' N_{j_t} F_t \tilde{X}_t + \tilde{X}_t' F_t' R_{j_t} F_t \tilde{X}_t. \end{aligned} \tag{10}$$

$$\tilde{A}_{j_{t+1}} \equiv \begin{bmatrix} A_{11j_{t+1}} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{B}_{j_{t+1}} \equiv \begin{bmatrix} A_{12j_{t+1}} & B_{1j_{t+1}} & 0 \\ 0 & 0 & I_{n_x} \end{bmatrix}, \tilde{C}_{j_t} \equiv \begin{bmatrix} C_{j_{t+1}} \\ 0 \end{bmatrix}.$$

2.3 Optimal policy and dynamics

- Solution: quasi-quadratic dual value function

$$\tilde{X}_t'\tilde{V}_{j_t}\tilde{X}_t + \tilde{w}_{j_t} \equiv \max_{\{\gamma_{t+\tau}\}_{\tau\geq 0}} \min_{\{x_{t+\tau}, i_{t+\tau}\}_{\tau\geq 0}} \mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^{\tau} \tilde{L}_{t+\tau} \qquad (j_t = 1, ..., n)$$

- Quasi-liner optimal policy function (also solves primal problem)

$$\begin{bmatrix} x_t \\ i_t \\ \Xi_t \end{bmatrix} = F_{j_t}\tilde{X}_t \equiv \begin{bmatrix} F_{xj_t} \\ F_{ij_t} \\ F_{\Xi j_t} \end{bmatrix} \tilde{X}_t \qquad (j_t = 1, ..., n)$$

- Bellman equation

$$\tilde{X}_t'\tilde{V}_{j_t}\tilde{X}_t + \tilde{w}_t = \max_{\gamma_t} \min_{(x_t, i_t)} \left\{ \tilde{X}_t' Q_{j_t}\tilde{X}_t + 2\tilde{X}_t' N_{j_t}\tilde{\imath}_t + \tilde{\imath}_t' R_{j_t}\tilde{\imath}_t + \delta \mathrm{E}_t(\tilde{X}_{t+1}'\tilde{V}_{j_{t+1}}\tilde{X}_{t+1} + \tilde{w}_{j_{t+1}}) \right\},$$

First-order condition w.r.t. $\tilde{\imath}_t$

$$\tilde{X}_t' N_t + \tilde{\imath}_t' R_t + \delta \tilde{X}_t' \mathrm{E}_t \tilde{A}_{t+1}' \tilde{V}_{t+1} \tilde{B}_{t+1} + \delta \tilde{\imath}_t' \mathrm{E}_t \tilde{B}_{t+1}' \tilde{V}_{t+1} \tilde{B}_{t+1} = 0,$$

$$J_t \tilde{\imath}_t + K_t \tilde{X}_t = 0,$$

$$J_t \equiv R_t + \delta \mathrm{E}_t \tilde{B}_{t+1}' \tilde{V}_{t+1} \tilde{B}_{t+1}, \qquad (11)$$

$$K_t \equiv N_t' + \delta \mathrm{E}_t \tilde{B}_{t+1}' \tilde{V}_{t+1} \tilde{A}_{t+1}. \qquad (12)$$

Optimal policy function

$$\tilde{\imath}_t = F_t \tilde{X}_t, \qquad (13)$$

$$F_t \equiv -J_t^{-1} K_t. \qquad (14)$$

Value function satisfies

$$\begin{aligned}\tilde{X}_t'\tilde{V}_t\tilde{X}_t &\equiv \tilde{X}_t'Q_t\tilde{X}_t + 2\tilde{X}_t'N_tF_t\tilde{X}_t + \tilde{X}_t'F_t'R_tF_t\tilde{X}_t \\ &\quad + \delta\tilde{X}_t'\mathrm{E}_t[(\tilde{A}_{t+1}' + F_t'\tilde{B}_{t+1}')\tilde{V}_{t+1}(\tilde{A}_{t+1} + \tilde{B}_{t+1}F_t)]\tilde{X}_t.\end{aligned}$$

$\{\tilde{V}_j\}$ solve coupled Riccati equations

$$\tilde{V}_j = Q_j + \delta\sum_{k=1}^{n} P_{jk}\tilde{A}_k'\tilde{V}_k\tilde{A}_k - K_j'J_j^{-1}K_j \qquad (j = 1, ..., n),$$

$\{\tilde{w}_j\}$ solve the equations

$$\tilde{w}_j = \delta\sum_k P_{jk}[\mathrm{tr}(\tilde{V}_k\tilde{C}_k\tilde{C}_k') + \tilde{w}_k].$$

- Algorithms in paper: Follow CFM in solving coupled Riccati equations via sequence of standard (uncoupled) LQR problems
- Primal value function

$$\tilde{X}_t'V_j\tilde{X}_t + w_j \equiv \min \mathrm{E}_t\sum_{\tau=0}^{\infty}\delta^\tau L_{t+\tau} \equiv \tilde{X}_t'\tilde{V}_j\tilde{X}_t + \tilde{w}_j - \frac{1}{\delta}\Xi_{t-1}'H_jF_{xj}\tilde{X}_t \qquad (j = 1, ..., n)$$

$$w_j = \tilde{w}_j \qquad (j = 1, ..., n)$$

- Dynamics: Transition from $(\tilde{X}_t, j_t)$ to $(\tilde{X}_{t+1}, j_{t+1})$

$$\begin{aligned}\tilde{X}_{t+1} &= M_{j_tj_{t+1}}\tilde{X}_t + \tilde{C}_{j_{t+1}}\varepsilon_{t+1}, \\ M_{j_tj_{t+1}} &\equiv \tilde{A}_{j_{t+1}} + \tilde{B}_{j_{t+1}}F_{j_t}\end{aligned}$$

For given ε_{t+1}, transition w/ probability $P_{j_tj_{t+1}}$

- Determines the optimal distribution of future $\tilde{X}_{t+\tau}$, $j_{t+\tau}$, and $\tilde{\imath}_{t+\tau}$ ($\tau \geq 1$) conditional on $(\tilde{X}_t, j_t)$.
  - Plot future means/medians/percentiles (**fan charts**)
  - Also future means/medians/percentiles for individual chains (**scenarios**) and, for comparison, CC/CE case

3 Interpretation of model uncertainty in our framework

Large variety of uncertainty configurations, approximating many different kinds of model uncertainty:

- i.i.d. and serially correlated random model coefficients (generalized Brainard-type uncertainty)
- Different structural models
  - Different variables, different number of leads and lags
  - Particular variable predetermined in one model and forward-looking in another
  - Backward- and forward-looking models
  - Backward- and forward-looking private-sector expectations
- Persistence of inflation, output, additive shocks.
- Variance of shocks, persistence of variance

- Different CB judgments about model coefficients and model uncertainty
  - n different *model* modes (different coefficients/coefficient variances/ coefficient persistence/shock variance (via C_j)) ($j_t = 1, ..., n$)
  - m different CB *judgment* modes (different CB information about model modes, including persistence) ($k_t = 1, ..., m$)
  - Transition matrix for model modes, depend on the judgment mode, $\tilde{P}(k_t)$
  - Transition matrix for the judgment modes independent of the model modes, P^0
  - Composite model-judgment mode (j_t, k_t), transition probability from model-judgment mode $(h, k)_t$ to $(j, l)_{t+1}$, $\tilde{P}(k)_{hj} P^0_{kl}$.

- Unobservable modes: Optimal policy depends on probability distribution of modes, p_t
- Incorporate CB judgment also about future additive deviations as in "MP w/ Judgment"
- etc., etc.
- Any relevant situation for a policymaker that cannot be approximated in our framework?
- Aside from dimensional and computational limitations, not terribly tight.
  – Examples to be shown have 9 state variables, 3 modes, solves in seconds. Examples with 10 states and 50 modes solves in minutes
  – Speed not crucial for policy optimization, more relevant for policy analysis and estimation

4 Examples

Empirical regime-switching models of US economy

– Backward-looking model (Rudebusch-Svensson, RS)

– Forward-looking model (Lindé)

4.1 An estimated backward-looking model (RS)

Serially correlated parameter variation, 3 modes, Bayesian Gibbs sampling methods (Kim and Nelson), comparing w/ constant-coefficient (CC) version of RS

$$\begin{aligned}\pi_{t+1} &= \sum_{\tau=0}^{2} \alpha_{\tau j}\pi_{t-\tau} + (1 - \sum_{\tau=0}^{2} \alpha_{\tau j}\pi_{t-3}) + \alpha_{3j}y_t + c_\pi \varepsilon_{\pi,t+1}, \\ y_{t+1} &= \beta_{1j}y_t + \beta_{2j}y_{t-1} + \beta_{3j}\left(\bar{\imath}_t - \bar{\pi}_t\right) + c_y\varepsilon_{y,t+1},\end{aligned} \quad (15)$$

$j \in \{1, 2, 3\}$,

$\bar{\imath}_t \equiv \sum_{\tau=0}^{3} i_{t-\tau}/4$, $\bar{\pi}_t \equiv \sum_{\tau=0}^{3} \pi_{t-\tau}/4$,

$\varepsilon_{\pi t}$, ε_{yt} i.i.d. $N(0, 1)$

Table 4.1: Estimates of the constant-parameter
and three-mode Rudebusch-Svensson model.

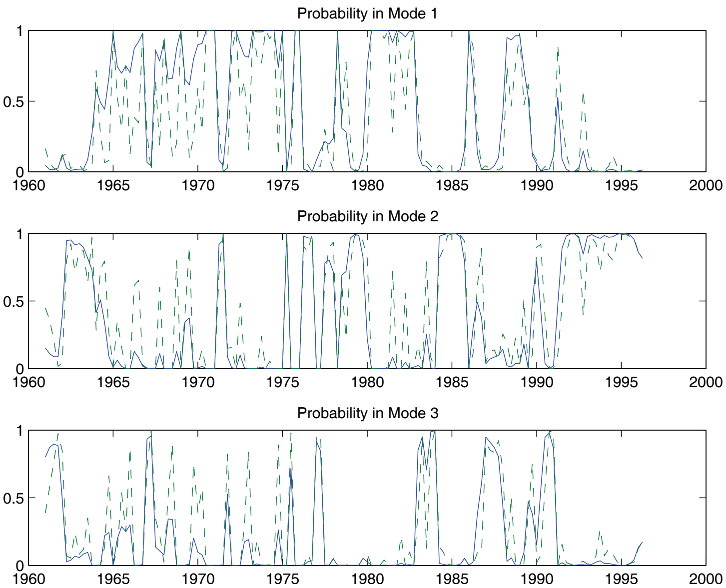
Parameter	Constant	Mode 1	Mode 2	Mode 3
α_0	0.6922	0.2402	0.4236	1.2387
α_1	−0.1033	0.1654	−0.2219	-0.6911
α_2	0.2786	1.0388	0.0714	0.5491
α_3	0.1021	0.1514	0.2755	−0.0304
β_1	1.1591	1.0015	1.0302	1.8943
β_2	−0.2521	−0.0853	−0.1069	−1.0312
β_3	−0.0990	−0.3244	0.0315	−0.1011
c_π	1.0090	1.5504	0.1798	0.1562
c_y	0.8190	1.2696	0.1447	0.2365

Substantial variation across modes: Volatility, slopes of Phillips and AD curve, signs

Transition matrix and stationary distribution

$$P = \begin{bmatrix} 0.8331 & 0.0921 & 0.0748 \\ 0.0305 & 0.9194 & 0.0501 \\ 0.0360 & 0.0541 & 0.9100 \end{bmatrix}, \qquad \bar{p} = \begin{bmatrix} 0.1652 & 0.4483 & 0.3866 \end{bmatrix}.$$

Figure 4.1: Estimated probabilities of being in the different modes.
Solid: smoothed (full-sample) inference.
Dashed: filtered (one-sided) inference.



Optimal policy RS model

- Period loss function

$$L_t = \pi_t^2 + \lambda y_t^2 + \nu(i_t - i_{t-1})^2 \tag{16}$$

$\lambda = 1$, $\nu = 0.2$, and $\delta = 1$

- Optimal policy function,

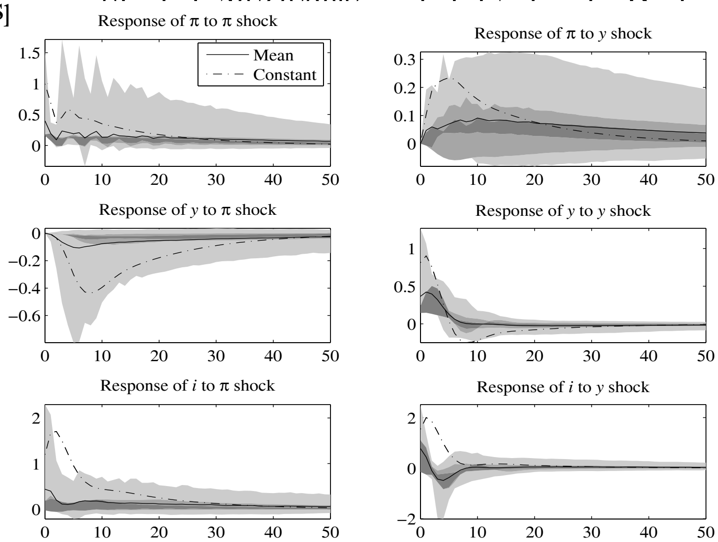
$$i_t = F_{ij} X_t \qquad (j = 1, 2, 3),$$

$$X_t \equiv (\pi_t, \pi_{t-1}, \pi_{t-2}, \pi_{t-3}, y_t, y_{t-1}, i_{t-1}, i_{t-2}, i_{t-3})'$$

Table 4.2: Optimal policy functions, RS model

Mode	π_t	π_{t-1}	π_{t-2}	π_{t-3}	y_t	y_{t-1}	i_{t-1}	i_{t-2}	i_{t-3}
Constant	0.9921	0.3465	0.4273	0.1381	1.7974	−0.4639	0.3713	−0.0899	−0.0456
Mode 1	1.4796	1.3130	1.0760	−0.2853	1.9834	−0.4890	−0.1723	−0.3271	−0.1834
Mode 2	−0.1510	−0.1739	−0.2132	−0.2077	−1.0595	−0.2824	0.3311	−0.0840	−0.0326
Mode 3	1.1526	0.0988	0.5878	0.0309	4.6475	−4.6851	−0.0205	−0.2364	−0.1245
Mean	0.6222	0.1771	0.3094	−0.1283	1.6490	−2.0182	0.1120	−0.1830	−0.0930

Figure 4.2: Unconditional impulse responses, RS model, optimal policy.
Solid: Mean res[...]l: CC/CE.



- Mean responses somewhat similar to CC/CE, but much dispersion. More similar in shape than magnitude.
- Wide persistent band for inflation shocks.
- Mean instrument-rate response **less aggressive** than CC/CE (Brainard!)

4.2 An estimated forward-looking model (Linde)

- Persistent parameter variation, 3 modes, Bayesian methods

$$\begin{aligned}
\pi_t &= \omega_{fj} E_t \pi_{t+1} + (1-\omega_{fj})\pi_{t-1} + \gamma_j y_t + c_\pi \varepsilon_{\pi t}, \\
y_t &= \beta_{fj} E_t y_{t+1} + (1-\beta_{fj}) \left[\beta_{yj} y_{t-1} + (1-\beta_{yj}) y_{t-2}\right] \\
&\quad - \beta_{rj} \left(i_t - E_t \pi_{t+1}\right) + c_y \varepsilon_{yt}, \\
i_t &= \left(1-\rho_{1j}-\rho_{2j}\right)\left(\gamma_{\pi j}\pi_t + \gamma_{yj} y_t\right) + \rho_{1j} i_{t-1} + \rho_{2j} i_{t-2} + c_i \varepsilon_{it}.
\end{aligned} \tag{17}$$

$j \in \{1,2,3\}$, $\varepsilon_{\pi t}$, ε_{yt}, ε_{it} i.i.d. $N(0,1)$

Table 4.3: Estimates of the constant-parameter and three-mode Lindé model.

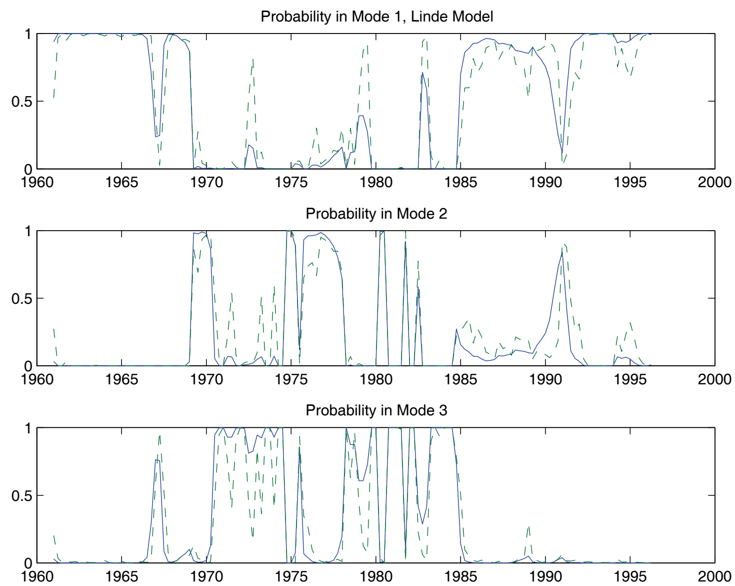
Parameter	Constant	Mode 1	Mode 2	Mode 3
ω_f	0.4908	0.4644	0.3380	0.3198
γ	0.0081	0.0112	0.0786	0.0312
β_f	0.4408	0.0889	0.2356	0.3911
β_r	0.0048	0.0396	0.1395	0.0000
β_y	1.1778	1.1119	1.1570	1.2312
ρ_1	0.9557	1.1486	0.8525	0.7967
ρ_2	−0.0673	−0.2340	−0.1172	0.0516
γ_π	1.3474	1.2439	−0.0643	2.3427
γ_y	0.7948	0.5799	0.9717	−0.3101
c_π	0.5923	0.4861	0.7232	0.9801
c_y	0.4162	0.4744	0.5083	0.6720
c_i	0.9918	0.2995	0.3930	1.2341

- Substantial variation across modes: Volatility, slopes

Transition matrix and stationary distribution

$$P = \begin{bmatrix} 0.9403 & 0.0340 & 0.0257 \\ 0.0625 & 0.8924 & 0.0451 \\ 0.0695 & 0.0576 & 0.8729 \end{bmatrix}, \qquad \bar{p} = \begin{bmatrix} 0.5229 & 0.2741 & 0.2030 \end{bmatrix}.$$

Figure 4.3: Estimated probabilities of being in the different modes.
Solid: Smoothed (full-sample) inference. Dashed: filtered (one-sided) inference.



- Optimal policy function,

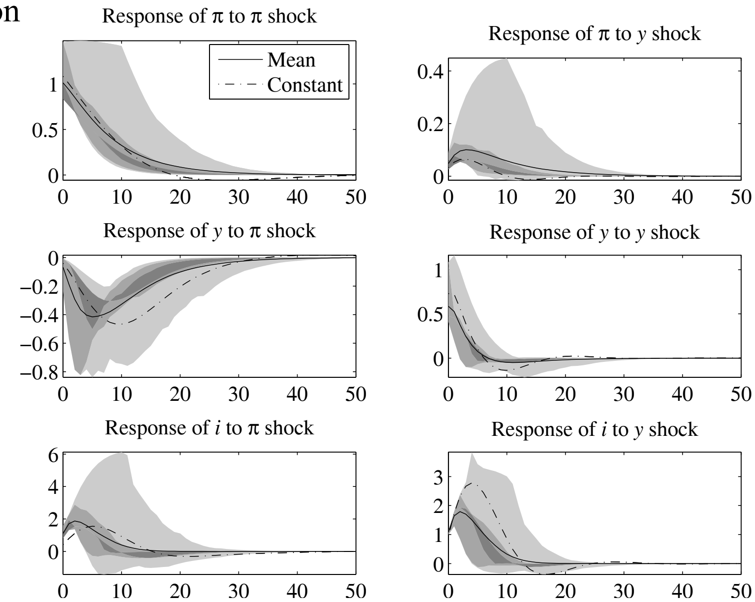
$$i_t = F_{ij}\tilde{X}_t,$$

$$\tilde{X}_t \equiv (\pi_{t-1}, y_{t-1}, y_{t-2}, \varepsilon_{\pi t}, \varepsilon_{yt}, \Xi_{\pi,t-1}, \Xi_{y,t-1})',$$

Table 4.3: Optimal policy functions, Lindé model.

Mode	π_{t-1}	y_{t-1}	y_{t-2}	i_{t-1}	$\varepsilon_{\pi t}$	ε_{yt}	$\Xi_{\pi,t-1}$	$\Xi_{y,t-1}$
Constant	0.3552	1.0714	−0.2231	0.7853	0.6975	2.2437	0.0024	0.0182
Mode 1	0.8915	2.0766	−0.2338	0.5962	1.6644	2.2929	0.0037	0.0066
Mode 2	1.4625	1.6985	−0.2666	0.3271	2.2092	2.2216	0.0090	0.0393
Mode 3	0.8348	0.7955	−0.2085	0.8016	1.2273	1.4812	0.0006	0.0021
Mean	1.0364	1.7130	−0.2376	0.5641	1.7249	2.1086	0.0045	0.0147

Figure 4.4: Unconditional impulse responses, Lindé model, optimal policy.
Solid: Mean respon /CE.



Mean responses somewhat similar to CC/CE case
Smaller overall impact of uncertainty compared to RS model
Mean i_t response more (less) aggressive to π_t (y_t) shocks
Substantial variation in responses, but die out after $\sim$30 quarters

5 Arbitrary time-varying policy rules

- Arbitrary time-varying policy rules (instrument rules/paths, targeting rules) (incl. constant instrument rate)
- Arbitrary time-varying policy rule during period $t = 0, 1, ..., T-1$,

$$i_t = F_{\tilde{X}tj_t}\tilde{X}_t + F_{xtj_t}x_t \qquad (0 \leq t \leq T-1), \tag{18}$$

$\tilde{X}_t \equiv (X_t', \Xi_{t-1}')'$

- Optimal policy function from period T on
  - $F_{xtj_t} \equiv 0$, *explicit* instrument rule
  - $F_{xtj_t} \not\equiv 0$ ($F_{xtj_t} \neq 0$ for some mode j_t with positive probability) *implicit* instrument rule (implementation problem)
- Also optimal policy rules in a given class of rules (Taylor-type rules)

7 Unobservable modes

- Practial importance: More relevant than observable modes
- **No learning**: p_t probability distribution in period t

$$p_{t+\tau} = (P')^\tau p_t$$

- W/o learning, still quasi linear-quadratic
- Work in progress: Learning, Bayesian updating of p_t, nonlinear

Revision of paper: Consider equation

$$\mathrm{E}_t H_{j_{t+1}} x_{t+1} = A_{21j_t} X_t + A_{22j_t} x_t - z_t + B_{2j_t} i_t. \tag{19}$$

Interpret left hand side as conditional on p_t, not on j_t
Practical to replace equation (19) by the two equivalent equations,

$$\mathrm{E}_t H_{j_{t+1}} x_{t+1} = z_t, \tag{20}$$

$$0 = A_{21j_t} X_t + A_{22j_t} x_t - z_t + B_{2j_t} i_t, \tag{21}$$

z_t additional forward-looking variables, z_t. Will depend on p_t, not j_t, whereas x_t will depend on j_t.
Solve x_t as a function of X_t, z_t, i_t, j_t, and ε_t

$$x_t = \tilde{x}(X_t, z_t, i_t, j_t) \equiv A_{22j_t}^{-1}(z_t - A_{21j_t} X_t - B_{2j_t} i_t). \tag{22}$$

For given j_t, this function is linear in X_t, z_t, i_t, and ε_t.
Primal period loss function: $L_t = L(X_t, x_t, i_t)$
Dual period loss function

$$\mathrm{E}_t \tilde{L}(\tilde{X}_t, z_t, i_t, \gamma_t, j_t) \equiv \sum_j p_{jt} \tilde{L}(\tilde{X}_t, z_t, i_t, \gamma_t, j),$$

$$\tilde{L}(\tilde{X}_t, z_t, i_t, \gamma_t, j_t) \equiv L[X_t, \tilde{x}(X_t, z_t, i_t, j_t), i_t, j_t] - \gamma_t' z_t + \Xi_{t-1}' \frac{1}{\delta} H_{j_t} \tilde{x}(X_t, z_t, i_t, j_t).$$

Dual optimization problem can be written

$$\tilde{X}_t'\tilde{V}(p_t)\tilde{X}_t + \tilde{w}(p_t) = \max_{\gamma_t} \min_{(z_t,i_t)} \mathrm{E}_t\{\tilde{L}(\tilde{X}_t, z_t, i_t, \gamma_t, j_t) + \delta[\tilde{X}_{t+1}'\tilde{V}(p_{t+1})\tilde{X}_{t+1} + \tilde{w}(p_{t+1})]\} \tag{23}$$

Transition equation

$$X_{t+1} = A_{11j_{t+1}}X_t + A_{12j_{t+1}}\tilde{x}(X_t, z_t, i_t, j_t) + B_{1j_{t+1}}i_t + C_{j_{t+1}}\varepsilon_{t+1}, \tag{24}$$

$$\Xi_t = \gamma_t, \tag{25}$$

$$p_{t+1} = P'p_t \tag{26}$$

The solution of the dual optimization problem is quasi-linear and can be written

$$z_t = F_z(p_t)\tilde{X}_t, \tag{27}$$

$$i_t = F_i(p_t)\tilde{X}_t, \tag{28}$$

$$\gamma_t = F_\gamma(p_t)\tilde{X}_t, \tag{29}$$

$$x_t = \tilde{x}[X_t, F_z(p_t)\tilde{X}_t, F_i(p_t)\tilde{X}_t, j_t] \equiv F_x(p_t)_{j_t}\tilde{X}_t. \tag{30}$$

The value function for the original problem, $\tilde{X}_t'V(p_t)\tilde{X}_t + w(p_t)$, with the period loss function $\mathrm{E}_tL(X_t, x_t, i_t, j_t)$ rather than $\mathrm{E}_t\tilde{L}(\tilde{X}_t, z_t, i_t, \gamma_t, j_t)$, satisfies

$$\tilde{X}_t'V(p_t)\tilde{X}_t \equiv \tilde{X}_t'\tilde{V}(p_t)\tilde{X}_t - \Xi_{t-1}'\frac{1}{\delta}\sum_j p_{jt}H_jF_x(p_t)_{j_t}\tilde{X}_t, \tag{31}$$

$$w(p_t) \equiv \tilde{w}(p_t).$$

Computing $F_i(p_t)$ and $V(p_t)$ for all feasible values of p_t requires standard function-approximation methods

Computing the functions for a particular value $p_t = \tilde{p}_t$ is straightforward

Details (not yet in paper for case with z_t)

$$L_t = \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}' \begin{bmatrix} Q_{11,j} & Q_{12,j} & N_{1,j} \\ Q_{12,j} & Q_{22,j} & N_{2,j} \\ N_{1,j}' & N_{2,j}' & R_j \end{bmatrix} \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}$$

$$\tilde{L}_t = L_t - \gamma_t'z_t + \Xi_{t-1}'\frac{1}{\delta}H_jx_t.$$

$$\begin{aligned} x_t &= A_{22,j}^{-1}z_t - A_{22,j}^{-1}A_{21,j}X_t - A_{22,j}^{-1}B_{2,j}i_t \\ &\equiv A_{xz,j}z_t + A_{xX,j}X_t + A_{xi,j}i_t, \end{aligned}$$

Suppressing time and date subscripts (all are t and j, respectively) the dual loss can be written explicitly as:

$$\begin{aligned} \tilde{L}_t &= X'\left(Q_{11} + A_{xX}'Q_{22}A_{xX} + 2A_{xX}'Q_{12}'\right)X + 2X'\left(N_1 + Q_{12}A_{xi} + A_{xX}'Q_{22}A_{xi} + A_{xX}'N_2\right)i \\ &\quad + 2z'(A_{xz}'Q_{12}' + A_{xz}'Q_{22}A_{xX})X + \Xi'\frac{1}{\delta}HA_{xX}X + \Xi'\frac{1}{\delta}HA_{xz}z + \Xi'\frac{1}{\delta}HA_{xi}i \\ &\quad - \gamma'z + z'(A_{xz}'Q_{22}A_{xz})z + i'(R + A_{xi}'Q_{22}A_{xi} + 2A_{xi}'N_2)i + 2z'(A_{xz}'N_2 + A_{xz}'Q_{22}A_{xi})i. \end{aligned}$$

$$\tilde{L}_t = \begin{bmatrix} \tilde{X}_t \\ \tilde{\imath}_t \end{bmatrix}' \begin{bmatrix} \tilde{Q}_j & \tilde{N}_j \\ \tilde{N}_j' & \tilde{R}_j \end{bmatrix} \begin{bmatrix} \tilde{X}_t \\ \tilde{\imath}_t \end{bmatrix}$$

$$
\begin{aligned}
\tilde{X}_{t+1} &= \tilde{A}_{j_t,j_{t+1}}\tilde{X}_t + \tilde{B}_{j_t,j_{t+1}}\tilde{\imath}_t + \tilde{C}_{j_t,j_{t+1}}\varepsilon_{t+1} \\
\tilde{A}_{j,k} &= \begin{bmatrix} A_{11,k} + A_{12,k}A_{xX,j} & 0 \\ 0 & 0 \end{bmatrix}, \\
\tilde{B}_{j,k} &= \begin{bmatrix} A_{12,k}A_{xz,j} & B_{1,k} + A_{12,k}A_{xi,j} & 0 \\ 0 & 0 & I \end{bmatrix}, \\
\tilde{C}_{j,k} &= \begin{bmatrix} C_k & A_{12,k}A_{xv,j} \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

$$
\tilde{X}_t'\tilde{V}(p_t)\tilde{X}_t + \tilde{w}(p_t) = \max_{\gamma_t} \min_{z_t,i_t} \sum_j p_{tj} \left\{ \begin{array}{l} \tilde{X}_t'\tilde{Q}_j\tilde{X}_t + 2\tilde{X}_t'\tilde{N}_j\tilde{\imath}_t + \tilde{\imath}_t'\tilde{R}_j\tilde{\imath}_t + \mathrm{tr}(\Lambda_j) \\ +\,\delta \sum_k P_{jk}[\tilde{X}_{t+1,jk}'\tilde{V}(P'p_t)\tilde{X}_{t+1,jk} + \tilde{w}(P'p_t)] \end{array} \right\},
$$

First-order condition with respect to $\tilde{\imath}_t$

$$
\sum_j p_{tj} \left[\tilde{X}_t'\tilde{N}_j + \tilde{\imath}_t'\tilde{R}_j + \delta \sum_k P_{jk}(\tilde{X}_t'\tilde{A}_{j,k}' + \tilde{\imath}_t'\tilde{B}_{j,k}')\tilde{V}(P'p_t)\tilde{B}_{j,k} \right] = 0,
$$

$$
\sum_j p_{tj} \left[\tilde{N}_j'\tilde{X}_t + \tilde{R}_j\tilde{\imath}_t + \delta \sum_k P_{jk}\tilde{B}_{j,k}'\tilde{V}(P'p_t)(\tilde{A}_{j,k}\tilde{X}_t + \tilde{B}_{j,k}\tilde{\imath}_t) \right] = 0.
$$

$$
J(p_t)\tilde{\imath}_t + K(p_t)\tilde{X}_t = 0, \tag{32}
$$

$$
J(p_t) \equiv \sum_j p_{tj} \left[\tilde{R}_j + \delta \sum_k P_{jk}\tilde{B}_{j,k}'\tilde{V}(P'p_t)\tilde{B}_{j,k} \right]
$$

$$
K(p_t) \equiv \sum_j p_{tj} \left[\tilde{N}_j' + \delta \sum_k P_{jk}\tilde{B}_{j,k}'\tilde{V}(P'p_t)\tilde{A}_{j,k} \right]
$$

$$
\tilde{\imath}_t = \tilde{F}(p_t)\tilde{X}_t
$$

$$
\tilde{F}(p_t) \equiv -J(p_t)^{-1}K(p_t).
$$

$$
\tilde{X}_t'\tilde{V}(p_t)\tilde{X}_t \equiv \sum_j p_{tj} \left\{ \begin{array}{l} \tilde{X}_t'\tilde{Q}_j\tilde{X}_t + 2\tilde{X}_t'\tilde{N}_j\tilde{F}(p_t)\tilde{X}_t + \tilde{X}_t'\tilde{F}(p_t)'\tilde{R}_j\tilde{F}(p_t)\tilde{X}_t \\ +\,\delta \sum_k P_{jk}\tilde{X}_t'[\tilde{A}_{j,k}' + \tilde{F}(p_t)'\tilde{B}_{j,k}']\tilde{V}(P'p_t)[\tilde{A}_{j,k} + \hat{B}_{j,k}\tilde{F}(p_t)]\tilde{X}_t \end{array} \right\}.
$$

Riccati equation for $\tilde{V}(p_t)$:

$$
\tilde{V}(p_t) = \sum_j p_{tj} \left\{ \begin{array}{l} \tilde{Q}_j + \tilde{N}_j\tilde{F}(p_t) + \tilde{F}(p_t)'\tilde{N}_j' + \tilde{F}(p_t)'\tilde{R}_j\tilde{F}(p_t) \\ +\,\delta \sum_k P_{jk}[\tilde{A}_{j,k}' + \tilde{F}(p_t)'\tilde{B}_{j,k}']\tilde{V}(P'p_t)[\tilde{A}_{j,k} + \tilde{B}_{j,k}\tilde{F}(p_t)] \end{array} \right\}.
$$

Algorithm (correction of appendix I.3 of Svensson-Williams 05)
Starting point: Assume observable modes $T+1$ periods ahead, in period $t+T+1$. Use previous solution $\tilde{F}_j$ and $\tilde{V}_j$ as starting point.
These matrices and horizon T known
Consider an iteration for $\tau = T, T-1, ..., 0$, which determines $\tilde{F}(p_t)$ and thereby $\tilde{V}(p_t)$ as a function of T.
Increase horizon T until $\tilde{F}(p_t)$ and $\tilde{V}(p_t)$ converges.

(1) $p_{t+\tau,t}$ for $\tau = 0, ..., T$ and given p_t is determined $p_{t+\tau,t} = (P')^\tau p_t$.
(2) Let $\tilde{V}^{T+1}$ denote the mode-dependent matrices $\tilde{V}_k$ ($k = 1, ..., n$) (or any arbitrary symmetric positive semidefinite matrix).
(3) For $\tau = T, T-1, ..., 0$, let the mode-*in*dependent matrices $\tilde{V}^\tau$ and F^τ be determined recursively by

$$J^\tau \equiv \sum_j p_{t+\tau,j} \left[\tilde{R}_j + \delta \sum_k P_{jk} \tilde{B}'_{j,k} \tilde{V}^{\tau+1} \tilde{B}_{j,k} \right],$$

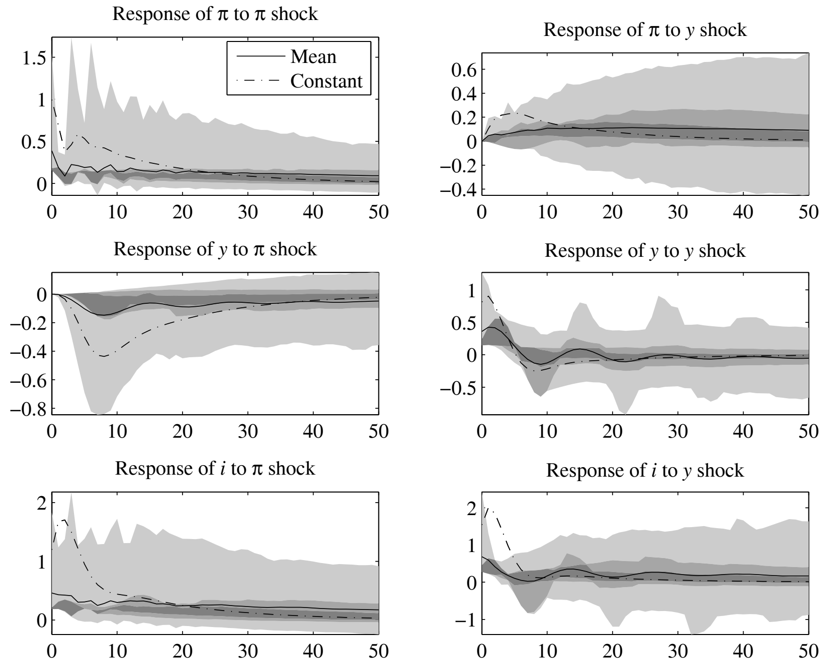
$$K^\tau \equiv \sum_j p_{t+\tau,j} \left[\tilde{N}'_j + \delta \sum_k P_{jk} \tilde{B}'_{j,k} \tilde{V}^{\tau+1} \tilde{A}_{j,k} \right]$$

$$\tilde{F}^\tau = -(J^\tau)^{-1} K^\tau,$$

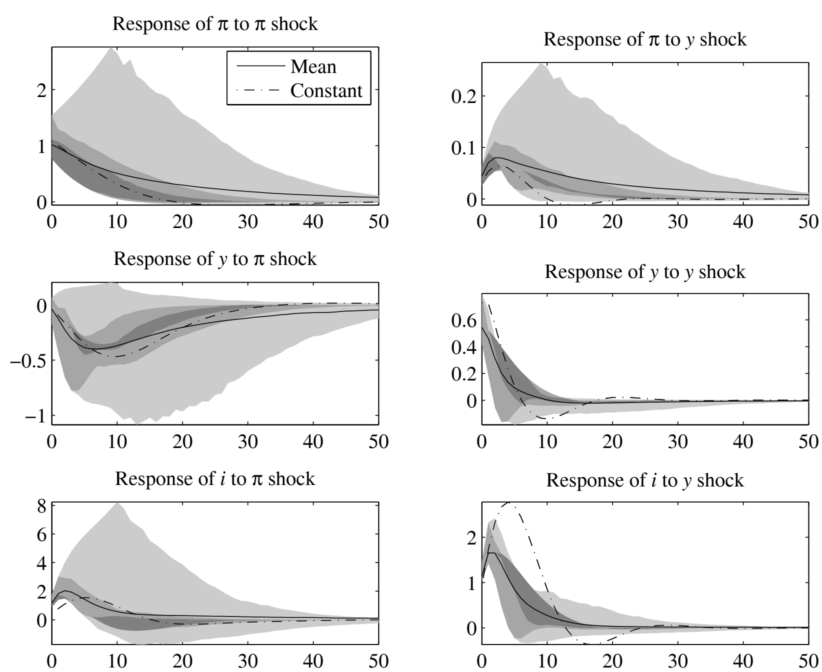
$$\tilde{V}^\tau = \sum_j p_{t+\tau,j} \left\{ \begin{array}{l} \tilde{Q}_j + \tilde{N}_j \tilde{F}^\tau + \tilde{F}^{\tau\prime} \tilde{N}'_j + \tilde{F}^{\tau\prime} \tilde{R}_j \tilde{F}^\tau \\ + \delta \sum_k P_{jk} [\tilde{A}'_{j,k} + \tilde{F}^{\tau\prime} \tilde{B}'_{j,k}] \tilde{V}^{\tau+1} [\tilde{A}_{j,k} + \tilde{B}_k \tilde{F}^\tau)] \end{array} \right\}.$$

Then $\tilde{F}(p_t) = \hat{F}^0$ and $\tilde{V}(p_t) = \tilde{V}^0$.
(4) Increase T , repeat from (1) until $\tilde{F}^0$ and $\tilde{V}^0$ have converged.

Unobservable RS: Impulse responses



Unobservable Lindé: Impulse responses



- RS: Mean policy responses much longer lasting
- Skewed distribution of responses. Most mass has small responses, but wide tails
- Not being able to respond to modes limits ability to stabilize economy
- Lindé: Wider tails for inflation shocks, tighter for output shocks
- Somewhat slower convergence, but not very sustained
- **Expectations** seem to matter in stabilization

8 Conclusions

- MJLQ framework flexible, powerful, yet tractable way of handling model uncertainty and non-CE
- Large variety of uncertainty configurations, also able to incorporate a large variety of CB judgment
- Extension to forward-looking variables via recursive saddlepoint method
- Straightforward to incorporate unobservable modes w/o learning. Work in progress on learning
- Provide advice on optimal policy for a large variety of uncertainty and judgment configurations
- Natural applications: put more **structure** on modes
  - Different structural models, or parameter uncertainty within models (as in LOWW Macroannual)
  - Uncertainty about driving processes: productivity slowdowns, volatility moderation, different fiscal policies

- LQ and CE vs. MJLQ and non-CE
  - LQ, CE, mean forecast targeting:
    - Mean forecast paths, i^t, X^t, x^t
    - Explicit optimal policy function not needed
    - Recursive formulation not needed
  - MJLQ, no CE, distribution forecast targeting:
    - Distribution forecast,
    - Distribution of future instrument rates
    - Recursive formulation and optimal policy function practical
    - Unobservable modes, time-varying but mode-independent policy