Problem Set 1

1. Consider the Cagan model

$$m_t - p_t = -\eta(p_{t+1|t} - p_t),$$

where m_t and p_t are logs of the money supply and price level, respectively, and $\eta > 0$. Solve for the non-bubble price-level solution for the following (log) money-supply processes:

(a) Before period 0, the log money supply is expected to follow

$$m_{t+\tau|t} = m \qquad t < 0, \tau \geq 0.$$

In period 0, the money supply is announced to be

$$\begin{aligned} m_t &= m && 0 \leq t < T_1 \\ m_t &= \bar{m} > m && T_1 \leq t < T_2 \\ m_t &= m && T_2 \leq t. \end{aligned}$$

(b) Before period 0, the log money supply is expected to follow

$$m_{t+\tau|t} = m \qquad t < 0, \tau \geq 0.$$

In period 0, the money supply is announced to be

$$\begin{aligned} m_t &= m && 0 \leq t < T_1 \\ m_t &= m + \mu(t - T_1) && T_1 \leq t < T_2 \\ m_t &= m + \mu(T_2 - T_1) && T_2 \leq t. \end{aligned}$$

(c) The log money supply fulfills

$$m_t = m + \alpha p_t.$$

For what values of α does a unique non-bubble solution exist?

2. Assume that money pays a constant positive interest rate, $\bar{\imath}$.

(a) Assume that a representative household holds money and nominal one-period bonds. Assume that the bonds are not discount bonds but coupon bonds. That is, a bond bought in period t has a face value of 1 unit of account and is worth $1 + i_t$ unit of accounts in the beginning of period $t + 1$ including interest. Specify the representative household's period and intertemporal budget constraint (for the intertemporal constraint, specify the conditions assumed).

(b) Assume that consumption, C_t, and real balances, M_t/P_t, enter the household's utility function. Derive a relation between the household's marginal rate of substitution of consumption for real balances and the nominal interest rate.

3. Assume that money pays zero interest rates. Let the household's utility function be

$$u(C,m) = \ln C + V(m),$$

where $m \equiv M/P$. Derive the money-demand equation when

(a)

$$V(m) = \begin{cases} 2ma - m^2 & (0 \leq m \leq a) \\ a^2 & (m > a) \end{cases},$$

(b)

$$V(m) = \frac{m^{1-1/\sigma}}{1-1/\sigma}.$$

(c) Specify any satiation level of real balances for the utility functions in (a) and (b).

(d) What happens to money demand in (a) and (b) when interest rates go to zero and infinity, respectively.

4. Consider two definitions of seignorage,)

$$\begin{aligned} S_t^1 &\equiv \frac{M_t - M_{t-1}}{P_t}, \\ S_t^2 &= \frac{i_t}{1+i_t}\frac{M_t}{P_t}. \end{aligned}$$

Let

$$d_{t,s} = \begin{cases} 1 & (s=t) \\ \prod_{\tau=t}^{s-1} \frac{1}{1+r_\tau} & (s>t) \end{cases}$$

Show that

$$\sum_{s=t}^{\infty} d_{t,s} S_s^1 = \sum_{s=t}^{\infty} d_{t,s} S_s^2 - \frac{M_{t-1}}{P_t}.$$

5. Recall that, for x normal, we have

$$\mathrm{E}[\exp x] = \exp\left\{\mathrm{E}[x] + \frac{1}{2}\mathrm{Var}\,[x]\right\}.$$

Show that, if x and y are jointly lognormal,

$$\begin{aligned} \mathrm{E}[x] &= \exp\left(\mathrm{E}[\ln x] + \frac{1}{2}\mathrm{Var}[\ln x]\right) \\ \mathrm{E}[xy] &= \mathrm{E}[x]\mathrm{E}[y]\exp\left(\mathrm{Cov}[\ln x, \ln y]\right). \end{aligned}$$

6. Let the intertemporal utility function in period t be

$$\mathrm{E}_t \sum_{s=t}^{\infty} \beta^{s-t} u(C_t, m_t),$$

where the period utility function is

$$u(C, m) = \frac{C^{1-1/\sigma} + m^{1-1/\sigma}}{1 - 1/\sigma}$$

and $m \equiv M/P$. Assume that, conditional on information available in period t, consumption growth, C_{t+1}/C_t, and gross inflation, P_{t+1}/P_t, are jointly lognormal.

(a) Derive an expression for the nominal interest rate in terms of the parameters of the intertemporal utility function and the joint probability distribution.

(b) Derive an expression for the inflation risk premium in the Fisher equation in terms of the same parameters.

7. Let

$$z_t = \rho z_{t-1} + \varepsilon_t,$$

where $|\rho| < 1$ and $\mathrm{E}_{t-1}\varepsilon_t = 0$.

(a) Solve the stochastic difference equation

$$\mathrm{E}_t[y_{t+2} - 2a y_{t+1} + b y_t] = z_t,$$

for y_t, $t \geq 0$, where $0 < b < 1$ and $2a > b + 1$.

(b) Solve the stochastic difference equation

$$\mathrm{E}_t[y_{t+1} - 2a y_t + b y_{t-1}] = z_t,$$

where $0 < b < 1$, $2a > b + 1$, and y_{t-1} is given.

8. Consider the problem

$$\min \mathrm{E}_t \sum_{s=t}^{\infty} (1 - \delta)\delta^{s-t} L_s,$$

where $0 < \delta < 1$ and

$$L_s = \frac{1}{2}[(\pi_s - \pi^*)^2 + \lambda x_s^2]$$

($\lambda \geq 0$), subject to

$$\pi_{s+1} = \pi_s + \alpha x_s + \varepsilon_{s+1},$$

for $s \geq t$, where $\alpha > 0$, $\mathrm{E}_s \varepsilon_{s+1} = 0$, π_t is given, and x_s is a control variable. Solve the problem and find the optimal reaction function,

$$x_s = \gamma_0 + \gamma \pi_s,$$

in two ways:

(a) by defining a value function $V(\pi_t) = k_0 + k_1 \pi_t + \frac{1}{2} k \pi_t^2$ (motivate why quadratic) and solving the Bellman equation, and

(b) by applying the method of Lagrange and solving the resulting difference equation.