Simple example 2, forward-looking model

Phillips curve

$$\pi_t - \pi = \delta(\pi_{t+1|t} - \pi) + \kappa x_t + u_t \tag{1}$$

where

$$\pi = \mathrm{E}[\pi_t]$$

"Cost-push shock"

$$u_t = \rho u_{t-1} + \varepsilon_t, \qquad 0 \le \rho < 1 \tag{2}$$

Aggregate demand

$$x_t = x_{t+1|t} - \sigma(i_t - \pi_{t+1|t} - \bar{r}) \tag{3}$$

Loss function

$$\mathrm{E}_t \sum_{s=t}^{\infty} (1-\delta)\delta^{s-t} L_s$$

$$L_s = \frac{1}{2}[(\pi_s - \pi^*)^2 + \lambda(x_s - x^*)^2]$$

State-space form

Predetermined variables, X_t: $u_t, 1$,

Forward-looking variables, $\tilde{x}_t$: π_t, x_t,

Instrument: i_t,

Target variables, Y_t: π_t, x_t,

Target levels, $\hat{Y}$: π^*, x^*

$$X_t \equiv \begin{bmatrix} u_t \\ 1 \end{bmatrix}, \qquad \tilde{x}_t \equiv \begin{bmatrix} \pi_t \\ x_t \end{bmatrix}, \qquad Y_t - \hat{Y} \equiv \begin{bmatrix} \pi_t - \pi^* \\ x_t - x^* \end{bmatrix}$$

$$\begin{aligned} \pi_{t+1|t} &= \pi + \frac{1}{\delta}(\pi_t - \pi) - \frac{\kappa}{\delta}x_t - \frac{1}{\delta}u_t \\ \sigma\pi_{t+1|t} + x_{t+1|t} &= x_t + \sigma i_t - \sigma\bar{r} \end{aligned}$$

$$\begin{bmatrix} u_{t+1} \\ 1 \\ \pi_{t+1|t} \\ \sigma\pi_{t+1|t} + x_{t+1|t} \end{bmatrix} \equiv \begin{bmatrix} X_{t+1} \\ H\tilde{x}_{t+1|t} \end{bmatrix} = \underbrace{\begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\delta} & \frac{\delta-1}{\delta}\pi & \frac{1}{\delta} & -\frac{\kappa}{\delta} \\ 0 & -\sigma\bar{r} & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} u_t \\ 1 \\ \pi_t \\ x_t \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \sigma \end{bmatrix}}_{B} i_t + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{C} \varepsilon_{t+1}$$

$$Y_t - \hat{Y} \equiv \begin{bmatrix} \pi_t - \pi^* \\ x_t - x^* \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\pi^* & 1 & 0 & 0 \\ 0 & -x^* & 0 & 1 & 0 \end{bmatrix}}_{D} \begin{bmatrix} u_t \\ 1 \\ \pi_t \\ x_t \\ i_t \end{bmatrix}$$

$$L_t = \frac{1}{2} \begin{bmatrix} \pi_t - \pi^* & x_t - x^* \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}}_{W} \begin{bmatrix} \pi_t - \pi^* \\ x_t - x^* \end{bmatrix} = \frac{1}{2} \begin{bmatrix} u_t & 1 & \pi_t & x_t & i_t \end{bmatrix} \tilde{W} \begin{bmatrix} u_t \\ 1 \\ \pi_t \\ x_t \\ i_t \end{bmatrix}$$

$$\tilde{W} \equiv D'WD$$

Simplify model (i_t not target variable, separate): Consider x_t the instrument, drop (3) and i_t
When $x_t, x_{t+1|t}, \pi_{t+1|t}$ have been determined, use (3) to find i_t

- Predetermined variables: $u_t, 1$
- Forward-looking variable: π_t
- Instrument: x_t
- Target variables: π_t, x_t

$$\begin{bmatrix} u_{t+1} \\ 1 \\ \pi_{t+1|t} \end{bmatrix} = \underbrace{\begin{bmatrix} \rho & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{\delta} & \frac{\delta-1}{\delta}\pi & \frac{1}{\delta} \end{bmatrix}}_{A} \begin{bmatrix} u_t \\ 1 \\ \pi_t \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -\frac{\kappa}{\delta} \end{bmatrix}}_{B} x_t + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{C} \varepsilon_{t+1}$$

$$Y_t - \hat{Y} \equiv \begin{bmatrix} \pi_t - \pi^* \\ x_t - x^* \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\pi^* & 1 & 0 \\ 0 & -x^* & 0 & 1 \end{bmatrix}}_{D} \begin{bmatrix} u_t \\ 1 \\ \pi_t \\ x_t \end{bmatrix}$$

$$L_t = \frac{1}{2} \begin{bmatrix} \pi_t - \pi^* & x_t - \pi^* \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}}_{W} \begin{bmatrix} \pi_t - \pi^* \\ x_t - x^* \end{bmatrix} = \frac{1}{2} \begin{bmatrix} u_t & 1 & \pi_t & x_t \end{bmatrix} \tilde{W} \begin{bmatrix} u_t \\ 1 \\ \pi_t \\ x_t \end{bmatrix}$$

$$\tilde{W} \equiv D'WD$$

• Optimal policy under commitment, starting in period $t = t_0$
Lagrangian

$$\mathcal{L}_{t_0} \equiv \mathrm{E}_{t_0} \sum_{t=t_0}^{\infty} (1-\delta)\delta^{t-t_0} \left\{ \frac{1}{2}[(\pi_t - \pi^*)^2 + \lambda(x_t - x^*)^2] + \Xi_t[\delta(\pi_{t+1} - \pi) + \kappa x_t + u_t - \pi_t + \pi \right. \tag{4}$$

First-order conditions ($t = t_0$):

$$\pi_{t_0} - \pi^* - \Xi_{t_0} = 0,$$
$$\lambda(x_{t_0} - x^*) + \kappa \Xi_{t_0} = 0,$$

($t \geq t_0 + 1$):

$$\pi_t - \pi^* - \Xi_t + \Xi_{t-1} = 0, \tag{5}$$
$$\lambda(x_t - x^*) + \kappa \Xi_t = 0 \tag{6}$$

Introduce

$$\Xi_{t_0 - 1} = 0 \tag{7}$$

Then (5), (6) and (7) are FOCs for $t \geq t_0$.

– Eliminate multipliers, "consolidated" first-order condition (targeting rule)
For $t = t_0$:

$$\pi_{t_0} - \pi^* + \frac{\lambda}{\kappa}(x_{t_0} - x^*) = 0$$

For $t \geq t_0 + 1$:

$$\pi_t - \pi^* + \frac{\lambda}{\kappa}(x_t - x_{t-1}) = 0 \tag{8}$$

Unconditional mean

$$\pi - \pi^* = \mathrm{E}[\pi_t] - \pi^* = -\frac{\lambda}{\kappa}(\mathrm{E}[x_t] - \mathrm{E}[x_{t-1}]) = 0 \tag{9}$$

Substitute (5) and (6) into (1), use (9): Difference equation for Ξ_t for $t \geq t_0$

$$\Xi_{t+1|t} - 2a\Xi_t + \frac{1}{\delta}\Xi_{t-1} = -\frac{1}{\delta}u_t - \frac{\kappa}{\delta}x^*, \tag{10}$$

$$2a \equiv 1 + \frac{1}{\delta} + \frac{\kappa^2}{\delta\lambda}$$

Initial condition (7) (Ξ_t predetermined variable)
(Alternative: Use (6) to eliminate Ξ_t; derive difference equation for x_t for $t \geq 0$, with "artifical" initial condition $x_{t_0-1} = x^*$.)

- One solution method (lag operators)

$$\Xi_{t+1|t} - 2a\Xi_t + \frac{1}{\delta}\Xi_{t-1} = -\frac{1}{\delta}u_t - \frac{\kappa}{\delta}x^*$$

Unconditional mean (assume stationary solution)

$$\begin{aligned}
\Xi - (1 + \frac{1}{\delta} + \frac{\kappa^2}{\delta\lambda})\Xi + \frac{1}{\delta}\Xi &= -\frac{\kappa}{\delta}x^* \\
\Xi &= \frac{\lambda}{\kappa}x^* \\
\tilde{\Xi}_t &\equiv \Xi_t - \Xi
\end{aligned}$$

$$\tilde{\Xi}_{t+1|t} - 2a\tilde{\Xi}_t + \frac{1}{\delta}\tilde{\Xi}_{t-1} = -\frac{1}{\delta}u_t \equiv A_t$$

$$\mathrm{E}_t\left[(\mathrm{L}^{-1} - 2a + \frac{1}{\delta}\mathrm{L})\tilde{\Xi}_t = A_t\right]$$

$$\mathrm{E}_t\left[(\mathrm{L}^{-2} - 2a\mathrm{L}^{-1} + \frac{1}{\delta})\mathrm{L}\tilde{\Xi}_t = A_t\right]$$

Characteristic equation,

$$\mu^2 - 2a\mu + \frac{1}{\delta} = 0, \tag{11}$$

- Two roots (eigenvalues), $\mu_1 = c$, $0 < c \equiv a - \sqrt{a^2 - \frac{1}{\delta}} < 1$, and $\mu_2 = \frac{1}{\delta c}$, such that $0 < \mu_1 = c < 1 < \frac{1}{\delta} < \mu_2 = \frac{1}{\delta c}$.

$$\mathrm{E}_t\left[(\mathrm{L}^{-1} - c)(\mathrm{L}^{-1} - \frac{1}{\delta c})\mathrm{L}\tilde{\Xi}_t = A_t\right]$$

$$\mathrm{E}_t\left[(1 - c\mathrm{L})(\mathrm{L}^{-1} - \frac{1}{\delta c})\tilde{\Xi}_t = A_t\right]$$

$$\mathrm{E}_t\left[(1 - c\mathrm{L})(1 - \delta c\mathrm{L}^{-1})\tilde{\Xi}_t = -\delta c A_t\right]$$

$$\mathrm{E}_t\left[(1 - c\mathrm{L})\tilde{\Xi}_t = -\delta c\frac{1}{1 - \delta c\mathrm{L}^{-1}}A_t = -\delta c\sum_{s=0}^{\infty}(\delta c)^s A_{t+s}\right]$$

$$\tilde{\Xi}_t - c\tilde{\Xi}_{t-1} = -\delta c\sum_{s=0}^{\infty}(\delta c)^s A_{t+s|t} = c\sum_{s=0}^{\infty}(\delta c)^s u_{t+s|t}$$

- Solution

$$\Xi_t - \Xi = c\sum_{s=0}^{\infty}(\delta c)^s u_{t+s|t} + c(\Xi_{t-1} - \Xi)$$
$$= \frac{c}{1-\delta c\rho}u_t + c(\Xi_{t-1} - \Xi) \quad (12)$$
$$= \frac{c}{1-\delta c\rho}\sum_{s=0}^{t-t_0} c^s u_{t-s} - c^{t+1-t_0}\Xi$$

Use (12) in (5) and (6):

$$\pi_t - \pi^* = \frac{c}{1-\delta c\rho}u_t - (1-c)(\Xi_{t-1} - \Xi) \quad (13)$$
$$= \frac{c}{1-\delta c\rho}\left[u_t - (1-c)\sum_{s=0}^{t-1-t_0} c^s u_{t-1-s}\right] + (1-c)c^{t-t_0}\Xi \quad (14)$$
$$x_t = -\frac{\kappa}{\lambda}\frac{c}{1-\delta c\rho}u_t - \frac{\kappa}{\lambda}c(\Xi_{t-1} - \Xi) \quad (15)$$
$$= -\frac{\kappa}{\lambda}\frac{c}{1-\delta c\rho}\sum_{s=0}^{t-t_0} c^s u_{t-s} + \frac{\kappa}{\lambda}c^{t+1-t_0}\Xi \quad (16)$$

- No average inflation bias, $\mathrm{E}[\pi_t] = \pi^*$
- History-dependence
- "t_0-optimality": Dependence on t_0, one-time surprise
  - Woodford 03, Svensson-Woodford 05: "optimality in timeless perspective" ($t_0 \to -\infty$)

- Interest rates

  Rewrite IS equation, substitute

$$i_t = \bar{r} + \pi_{t+1|t} + \frac{1}{\sigma}(x_{t+1|t} - x_t) \tag{17}$$

  Combine with (8)

$$\begin{aligned} i_t &= \bar{r} + \pi_{t+1|t} - \frac{\kappa}{\lambda\sigma}(\pi_{t+1|t} - \pi^*) \\ &= \bar{r} + \pi^* + (1 - \frac{\kappa}{\lambda\sigma})(\pi_{t+1|t} - \pi^*) \\ &= \bar{r} + \pi^* + (1 - \frac{\kappa}{\lambda\sigma})\left[\frac{c}{1-\delta c\rho}\rho u_t - (1-c)(\Xi_t - \Xi)\right] \\ &= \bar{r} + \pi^* + (1 - \frac{\kappa}{\lambda\sigma})c\left[\frac{\rho - (1-c)}{1-\delta c\rho}u_t - (1-c)(\Xi_{t-1} - \Xi)\right] \end{aligned}$$

  – Indeterminacy problems, i_t exogenous, see Svensson-Woodford 05

- Discretion

  Optimizing in period $t = t_0$

  First-order conditions for $t = t_0$:

$$\begin{aligned} \pi_{t_0} - \pi^* - \Xi_{t_0} &= 0, \\ \lambda(x_{t_0} - x^*) + \kappa\Xi_{t_0} &= 0, \end{aligned}$$

  For $t \geq t_0 + 1$:

$$\pi_t - \pi^* - \Xi_t + \Xi_{t-1} = 0, \tag{18}$$

$$\lambda(x_t - x^*) + \kappa\Xi_t = 0 \tag{19}$$

  Optimizing in period $t = t_0 + 1$

  First-order condition for $t = t_0 + 1$:

$$\begin{aligned} \pi_{t_0+1} - \pi^* - \Xi_{t_0+1} &= 0, \\ \lambda(x_{t_0+1} - x^*) + \kappa\Xi_{t_0+1} &= 0, \end{aligned}$$

  For $t \geq t_0 + 2$:

$$\pi_t - \pi^* - \Xi_t + \Xi_{t-1} = 0, \tag{20}$$

$$\lambda(x_t - x^*) + \kappa\Xi_t = 0 \tag{21}$$

  – “Surprise” in period t. Reoptimize in period $t+1$, “surprise” also in period $t+1$.

Anticipate first-order conditions for all $t \geq t_0$:

$$\pi_t - \pi^* - \Xi_t \;=\; 0, \tag{22}$$

$$\lambda(x_t - x^*) + \kappa \Xi_t \;=\; 0, \tag{23}$$

Eliminate Ξ_t, consolidated first-order condition

$$\pi_t - \pi^* + \frac{\lambda}{\kappa}(x_t - x^*) = 0 \tag{24}$$

– Steady state

$$\pi = \mathrm{E}[\pi_t] = \pi^* - \frac{\lambda}{\kappa}(\mathrm{E}[x_t] - x^*) = \pi^* + \frac{\lambda}{\kappa}x^* \tag{25}$$

Average inflation bias, $\mathrm{E}[\pi_t] > \pi^*$, if $x^* > 0$

Use (24) and (25) to eliminate π_t in (1)

$$x_t = \frac{\delta\lambda}{\lambda + \kappa^2}x_{t+1|t} - \frac{\kappa}{\lambda + \kappa^2}u_t$$

Solve forward

$$\begin{aligned} x_t \;&=\; -\frac{\kappa}{\lambda + \kappa^2}\sum_{s=0}^{\infty}\left(\frac{\delta\lambda}{\lambda + \kappa^2}\right)^s u_{t+s|t} \\ &=\; -\frac{\kappa}{\kappa^2 + \lambda(1 - \delta\rho)}u_t, \end{aligned}$$

Resulting π_t, i_t

$$\pi_t - \pi^* = \frac{\lambda}{\kappa}x^* + \frac{\lambda}{\kappa^2 + \lambda(1 - \delta\rho)}u_t$$

$$\begin{aligned} i_t \;&=\; \bar{r} + \pi_{t+1|t} + \frac{1}{\sigma}(x_{t+1|t} - x_t) \\ &=\; \bar{r} + \pi^* + \frac{\lambda}{\kappa}x^* + \frac{\lambda}{\kappa^2 + \lambda(1 - \delta\rho)}\rho u_t \\ &\quad + \frac{1}{\sigma}(-\frac{\kappa}{\kappa^2 + \lambda(1 - \delta\rho)})(\rho u_t - u_t) \\ &=\; \bar{r} + \pi^* + \frac{\lambda}{\kappa}x^* + \frac{\lambda\rho + (1 - \rho)\kappa/\sigma}{\kappa^2 + \lambda(1 - \delta\rho)}u_t \end{aligned}$$

Corresponds to inefficient discretion equilibrium

– Average inflation bias, $\pi > \pi^*$, if $x^* > 0$

– Stabilization bias, different response coefficients to u_t

– No history-dependence

- Discretion, alternative: Follow dynamic-programming algorithm ("Optimization...")

$$V(u_t) \equiv \min_{x_t}\{\frac{1}{2}[(\pi_t - \pi^*)^2 + \lambda(x_t - x^*)^2] + \delta \mathrm{E}_t V(u_{t+1})\}$$

subject to (1) and

$$\pi_{t+1} = g_0 + g_1 u_{t+1}, \tag{26}$$

where

$$V(u_t) \equiv v_0 + v_1 u_t + \frac{1}{2} v_2 u_t^2$$

and where the coefficients g_0, g_1 and v_0, v_1, v_2 remain to be determined.
Note simple case since essential predetermined variables are exogenous (not generally the case).
By (1) and (26), we have

$$\pi_t - \pi = \kappa x_t + \text{ exogenous.}$$

Hence, simple first-order condition

$$\kappa(\pi_t - \pi^*) + \lambda(x_t - x^*) = 0. \tag{27}$$

Combine with (1), (2), and (27), solve for

$$\begin{aligned} x_t &= f_0 + f_1 u_t \\ \pi_t &= g_0 + g_1 u_t, \end{aligned}$$

and identify f_0, f_1, g_0, g_1.

- Suppose

$$\bar{r}_t = \bar{r} + \omega(\bar{r}_{t-1} - \bar{r}) + \eta_t, \qquad 0 < \omega < 1.$$

Does not affect π_t, x_t, only i_t (as long as i_t is not a target variable)

- Suppose current inflation and the output gap is predetermined and determined one period ahead:

$$\begin{aligned}\pi_{t+1|t} - \pi &= \delta(\pi_{t+2|t} - \pi) + \kappa x_{t+1|t} + u_{t+1|t} \\ x_{t+1|t} &= x_{t+2|t} - \sigma(i_{t+1|t} - \pi_{t+2|t} - \bar{r})\end{aligned}$$

Assume same loss function as before.
Assume

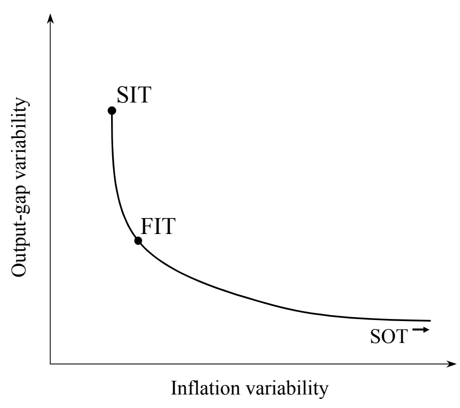
$$\begin{aligned}\pi_t &= \pi_{t|t-1} + u_t - u_{t|t-1} \\ x_t &= x_{t|t-1} + \eta_t\end{aligned}$$

Introduce

$$\begin{aligned}\tilde{\pi}_t &\equiv \pi_{t+1|t} \\ \tilde{x}_t &\equiv x_{t+1|t} \\ \tilde{u}_t &\equiv u_{t+1|t} \\ \tilde{\imath}_t &\equiv i_{t+1|t}\end{aligned}$$

Solve in terms of $\tilde{\pi}_t$, $\tilde{x}_t$, $\tilde{u}_t$ and $\tilde{\imath}_t$, then infer π_t, x_t and i_t (assume $i_t = i_{t|t-1} = \tilde{\imath}_{t-1}$ if i_t not target variable)

- "Taylor curves" (distinct from "Taylor rule"): $\mathrm{Var}[\pi_t]$ and $\mathrm{Var}[x_t]$ as function of $\lambda \geq 0$ (recall $c = c(\lambda)$)



Note

$$\lim_{\delta \to 1} (1-\delta) \mathrm{E}_t \sum_{s=t}^{\infty} \delta^{s-t} L_s = \mathrm{E}[L_t]$$

$$\begin{aligned}\mathrm{E}[L_t] &= \frac{1}{2}\mathrm{E}[(\pi_t - \pi^*)^2 + \lambda(x_t - x^*)^2] \\ &= \frac{1}{2}\{\mathrm{Var}[\pi_t] + (\mathrm{E}[\pi_t] - \pi^*)^2 + \lambda \mathrm{Var}[x_t] + \lambda(\mathrm{E}[x_t] - x^*)^2\}\end{aligned}$$

In situations without inflation bias and $x^* = 0$,

$$\mathrm{E}[L_t] = \frac{1}{2}(\mathrm{Var}[\pi_t] + \lambda \mathrm{Var}[x_t])$$

Under commitment, $\delta \approx 1$, $\mathrm{Var}[\pi_t]$ increasing in λ, $\mathrm{Var}[x_t]$ decreasing in λ
Compute $\mathrm{Var}[\pi_t]$ and $\mathrm{Var}[x_t]$ above, under commitment and discretion, as function of λ, for given ρ and σ_ε^2.

- Sum up
  - Predetermined and forward-looking variables
  - State-space form
  - Solution model
  - Special case here: Predetermined variables exogenous
  - Discretion/commitment equilibria different
    - Average inflation bias
    - Stabilization bias (response coefficients)
    - History-dependence (response to lagged predetermined variables)
  - Taylor curves