Last time

- Cagan model,

$$m_t - p_t = -\eta(p_{t+1|t} - p_t).$$

Solution (weighted average)

$$p_t = \frac{1}{1+\eta}\sum_{\tau=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{\tau} m_{t+\tau|t}.$$

Assumptions:
Convergence of series:

$$\left|\lim_{T\to\infty}\sum_{\tau=0}^{T-1}\left(\frac{\eta}{1+\eta}\right)^{\tau}\mathrm{E}_t m_{t+\tau}\right| < \infty,$$

which requires

$$\lim_{T\to\infty}\left(\frac{\eta}{1+\eta}\right)^{T}\mathrm{E}_t m_{t+T} = 0.$$

Absence of bubble solutions:

$$\lim_{T\to\infty}\left(\frac{\eta}{1+\eta}\right)^{T}\mathrm{E}_t p_{t+T} = 0.$$

Why assume absence of bubble solutions?

- Implication

$$\lim_{T\to\infty}\left(\frac{\eta}{1+\eta}\right)^{T}\mathrm{E}_t(m_{t+T} - p_{t+T}) = 0$$

  p_t does not diverge too far from m_t (still, considerable divergence possible)
- Simplest solution, focal point
- Microfoundations might exclude bubble solutions.

Empirical support of rational expectations (and this model)? Mixed

- Inherent difficulty: Dependence on conditional distributions rather than actual realizations. Joint test of RE and model
- Sargent 93: Sudden inflation stabilization after policy reform. But not sudden change in real balances.
- Expectations formation complex: Information and news other than realized inflation seem to matter.

2. An optimizing model of money demand and Friedmans's rule

- Cagan model simple empirical generalization. Microfoundations?
  - Is the Cagan model "structural"?
  - Other determinants of money demand (than $p_{t+1|t} - p_t$)?
  - Bubble solutions?
  - Integrate with other aspects of macro model, general equilibrium
  - Welfare analysis of monetary policy
- Sidrauski 67 - Brock 74 model
  - Representative household
  - Money modeled as financial asset, except
    - Fixed nominal return at zero (zero not essential)
    - Supplies "liquidity services" (by facilitating transactions), modeled as service flow from durable good
    - "Money in utility function"
      Other alternatives:
      - Transactions costs, transactions time
      - Cash in advance
      - Overlapping generations (money is only store of value)
  - Perfect foresight (at first)

- Household preferences

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} \left[u\left(C_s, \frac{M_s}{P_s}\right) + v(G_s) \right]$$

$0 < \beta < 1$, $u(C, m)$ concave, $C \geq 0$, $m \equiv M/P \geq 0$, $u_C > 0$, $u_m \geq 0$ (> 0 up to some possible satiation level $\bar{m}$, may depend on C), *real* balances; $v(G)$ additively separable utility of government consumption G

  - $u(C_s, \frac{M_s}{P_s})$ indirect utility function
    Simple transactions time interpretation,

$$u_t = \alpha \log C_t + (1-\alpha) \log \left(\bar{L} - L_t\right).$$

Leisure $\bar{L} - L_t$, L_t time spent in transations:

$$\bar{L} - L_t = A \left(\frac{M_t/P_t}{C_t}\right)^{\varepsilon}$$

$$\begin{aligned} u_t &= \alpha \log C_t + (1-\alpha) \log \left(\bar{L} - L_t\right) \\ &= \left[\alpha - \varepsilon(1-\alpha)\right] \log C_t + \varepsilon (1-\alpha) \log \frac{M_t}{P_t} + (1-\alpha) \log a \;\equiv\; u(C_t, \frac{M_t}{P_t}) \end{aligned}$$

  - u increasing in M/P corresponds to liquidity services from real balances (else not coexist with riskless short nominal bonds with positive i_t).

- Household period budget constraint (=, for simplicity)

Nominal:

$$P_tC_t + M_t + \frac{1}{1+i_t}B_t + P_t\frac{1}{1+r_t}b_t = P_tY_t - P_tT_t + M_{t-1} + B_{t-1} + P_tb_{t-1}$$

B_t (notation different from Obstfeld-Rogoff) denotes quantity of one-period nominal (riskless) bonds, each pays the equivalent of one unit of money in period $t+1$, nominal price $\frac{1}{1+i_t}$ in period t; b_t denotes quantity of one-period real bonds, each pays the equivalent of one consumption good in period $t+1$, real price $\frac{1}{1+r_t}$ in period t.

Real:

$$C_t + \frac{M_t}{P_t} + \frac{1}{P_t}\frac{1}{1+i_t}B_t + \frac{1}{1+r_t}b_t = Y_t - T_t + \frac{M_{t-1}}{P_t} + \frac{B_{t-1}}{P_t} + b_{t-1}$$

- First-order conditions for optimum

$$\max_{\{b_s,B_s,M_s\}_{s=t}^{\infty}} \sum_{s=t}^{\infty}\beta^{s-t}\left[u\left(Y_s - T_s + \frac{M_{s-1}}{P_s} + \frac{B_{s-1}}{P_s} + b_{s-1} - \frac{M_s}{P_s} - \frac{1}{P_s}\frac{1}{1+i_s}B_s - \frac{1}{1+r_s}b_s, \frac{M_s}{P_s}\right) + v(G_s)\right]$$

b_t:

$$\frac{1}{1+r_t}u_C\left(C_{t,}\frac{M_t}{P_t}\right) = \beta u_C\left(C_{t+1},\frac{M_{t+1}}{P_{t+1}}\right) \tag{1}$$

B_t:

$$\frac{1}{1+i_t}\frac{1}{P_t}u_C\left(C_{t,}\frac{M_t}{P_t}\right) = \frac{1}{P_{t+1}}\beta u_C\left(C_{t+1},\frac{M_{t+1}}{P_{t+1}}\right) \tag{2}$$

M_t:

$$\frac{1}{P_t}u_C\left(C_t,\frac{M_t}{P_t}\right) = \frac{1}{P_t}u_{M/P}\left(C_t,\frac{M_t}{P_t}\right) + \frac{1}{P_{t+1}}\beta u_C\left(C_{t+1},\frac{M_{t+1}}{P_{t+1}}\right) \tag{3}$$

- Rewrite as "asset price" conditions:

Real price (in terms of consumption good) of real bond

$$\frac{1}{1+r_t} = \frac{\beta u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}\right)}{u_C\left(C_{t,} \frac{M_t}{P_t}\right)} \tag{4}$$

(Right side $\equiv -$ MRS of C_t for C_{t+1})

Nominal price (in terms of money) of nominal bond

$$\frac{1}{1+i_t} = \frac{\beta u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}\right)/P_{t+1}}{u_C\left(C_{t,} \frac{M_t}{P_t}\right)/P_t} \tag{5}$$

(Right side $\equiv -$ MRS of P_tC_t at t for $P_{t+1}C_{t+1}$ at $t+1$)

Real price of money

$$\frac{1}{P_t} = \frac{u_{M/P}\left(C_t, \frac{M_t}{P_t}\right)/P_t}{u_C\left(C_t, \frac{M_t}{P_t}\right)} + \frac{\beta u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}\right)/P_{t+1}}{u_C\left(C_t, \frac{M_t}{P_t}\right)} \tag{6}$$

(Right side $= -$ MRS of C_t for M_t $-$ MRS of C_t for $P_{t+1}C_{t+1}$)

$$\frac{1}{P_t} = \frac{\beta u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}\right)/P_{t+1}}{u_C\left(C_t, \frac{M_t}{P_t}\right) - u_{M/P}\left(C_t, \frac{M_t}{P_t}\right)} \tag{7}$$

($u_{M/P} < u_C$)

From (4) and (5), Fisher equation under perfect foresight

$$\frac{1}{1+i_t} = \frac{P_t/P_{t+1}}{1+r_t} \tag{8}$$

From (5) and (6),

$$\frac{u_{M/P}\left(C_t, \frac{M_t}{P_t}\right)}{u_C\left(C_t, \frac{M_t}{P_t}\right)} = \frac{i_t}{1+i_t} \tag{9}$$

Left side: $-$MRS of consumption for real balances

$$\left.\frac{\mathrm{d}C_t}{\mathrm{d}(M_t/P_t)}\right|_{du_t=0} = -\frac{u_{M/P}\left(C_t, \frac{M_t}{P_t}\right)}{u_C\left(C_t, \frac{M_t}{P_t}\right)}$$

What is the right side? $-$MRT?

Introduce total nominal assets in the beginning of period $t+1$

$$W_{t+1} = M_t + B_t + P_{t+1} b_t \tag{10}$$

Note that (using (8))

$$\begin{aligned} M_t + \frac{1}{1+i_t} B_t + \frac{1}{1+r_t} P_t b_t &= \frac{i_t}{1+i_t} M_t + \frac{1}{1+i_t}(M_t + B_t) + \frac{1}{1+i_t} P_{t+1} b_t \\ &= \frac{i_t}{1+i_t} M_t + \frac{1}{1+i_t} W_{t+1} \end{aligned}$$

Rewrite the period budget constraint in nominal form,

$$P_t C_t + \frac{i_t}{1+i_t} M_t + \frac{1}{1+i_t} W_{t+1} = P_t Y_t - P_t T_t + W_t, \tag{11}$$

and in real form,

$$C_t + \frac{i_t}{1+i_t} \frac{M_t}{P_t} + \frac{1}{1+i_t} \frac{W_{t+1}}{P_t} = Y_t - T_t + \frac{W_t}{P_t}. \tag{12}$$

The term $\frac{i_t}{1+i_t}$ can be interpreted as the opportunity cost in period t of holding real balances (present value in period t of interest lost in period $t+1$, compared to holding nominal bonds). MRT of real balances into consumption,

$$\left. \frac{\mathrm{d}C_t}{\mathrm{d}(M_t/P_t)} \right|_{(12),\mathrm{d}(W_{t+1}/P_t)=0} = -\frac{i_t}{1+i_t}$$

- Intertemporal budget constraint (under perfect foresight). Multiply (12) by real (market) discount factor (present real value in period t of one unit real value in period s, $s \geq t$)

$$\begin{aligned} d_{t,t} &= 1 \\ d_{t,s} &\equiv \prod_{\tau=t}^{s-1} \frac{1}{1+r_\tau} = \frac{1}{(1+r_t)(1+r_{t+1})...(1+r_{s-1})} \quad (s > t) \end{aligned}$$

and sum (12) from $s = t$ to $T > t$, using (8):

$$\sum_{s=t}^{T} d_{t,s} \left(C_s + \frac{i_s}{1+i_s} \frac{M_s}{P_s} \right) + d_{t,T} \frac{1}{1+i_T} \frac{W_{T+1}}{P_T} = \sum_{s=t}^{T} d_{t,s}(Y_s - T_s) + \frac{W_t}{P_t} \tag{13}$$

Assume *no-Ponzi condition*,

$$\sum_{s=t}^{\infty} d_{t,s}(Y_s - T_s) + \frac{W_t}{P_t} \geq 0, \qquad \text{for all } t,$$

where we assume

$$\sum_{s=t}^{\infty} d_{t,s}(Y_s - T_s) < \infty.$$

Hence,

$$d_{t,T} \frac{1}{1+i_T} \frac{W_{T+1}}{P_T} = d_{t,T} \frac{1}{1+r_T} \frac{W_{T+1}}{P_{T+1}} \geq -d_{t,T+1} \sum_{s=T+1}^{\infty} d_{T+1,s}(Y_s - T_s),$$

and, used in (13),

$$\sum_{s=t}^{T} d_{t,s} \left(C_s + \frac{i_s}{1+i_s} \frac{M_s}{P_s} \right) \leq \sum_{s=t}^{\infty} d_{t,s}(Y_s - T_s) + \frac{W_t}{P_t} \tag{13a}$$

(LHS bounded)

Assume *transversality condition* (optimality condition, distinct from no-Ponzi condition),

$$\lim_{T\to\infty} d_{t,T}\frac{1}{1+i_T}\frac{W_{T+1}}{P_T} = 0. \tag{14}$$

Then, letting $T \to \infty$ in (13) gives the real intertemporal budget constraint,

$$\sum_{s=t}^{\infty} d_{t,s}\left(C_s + \frac{i_s}{1+i_s}\frac{M_s}{P_s}\right) = \sum_{s=t}^{\infty} d_{t,s}(Y_s - T_s) + \frac{W_t}{P_t}. \tag{15}$$

(LHS converges)
Note that

$$\sum_{s=t}^{\infty} d_{t,s}\left(C_s + \frac{i_s}{1+i_s}\frac{M_s}{P_s}\right) < \sum_{s=t}^{\infty} d_{t,s}(Y_s - T_s) + \frac{W_t}{P_t}$$

cannot be optimal; therefore (14) is an optimality condition.
The nominal (market) discount factor

$$\begin{aligned} D_{t,t} &= 1 \\ D_{t,s} &\equiv \prod_{\tau=0}^{s-1}\frac{1}{1+i_\tau} \quad (s>t) \end{aligned}$$

and the nominal period budget constraint (11) results in the nominal intertemporal budget constraint

$$\sum_{s=t}^{\infty} D_{t,s}\left(P_s C_s + \frac{i_s}{1+i_s}M_s\right) = \sum_{s=t}^{\infty} D_{t,s}P_s(Y_s - T_s) + W_t. \tag{16}$$

- Money demand. Solve (9) for $\frac{M_t}{P_t}$:

$$\frac{M_t}{P_t} = L(C_t, i_t)$$

  – Isoelastic utility function

$$u\left(C, \frac{M}{P}\right) = \frac{\left[C^\gamma (M/P)^{1-\gamma}\right]^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}}$$

$$\frac{M_t}{P_t} = \frac{1-\gamma}{\gamma}C_t\frac{1+i_t}{i_t}$$

  – CES utility function

$$u\left(C, \frac{M}{P}\right) = \frac{\left\{\left[\gamma^{\frac{1}{\eta}}C^{1-\frac{1}{\eta}} + (1-\gamma)^{\frac{1}{\eta}}\left(\frac{M}{P}\right)^{1-\frac{1}{\eta}}\right]^{\frac{\eta}{\eta-1}}\right\}^{1-\frac{1}{\sigma}} - 1}{1-\frac{1}{\sigma}}$$

$$\frac{M_t}{P_t} = \frac{1-\gamma}{\gamma}C_t\left(\frac{i_t}{1+i_t}\right)^{-\eta}$$

  (Note $\eta \to 1$!)
  Not identical to Cagan's:

$$\begin{aligned} \frac{M_t}{P_t} &\sim A(1+i_t)^{-\eta} \\ \log M_t - \log P_t &= \log A - \eta\log(1+i_t) \\ &= \log A - \eta[\log(1+r) + \log P_{t+1|t} - \log P_t] \end{aligned}$$

- Government budget constraint (for simplicity, drop real bonds)
  Consolidated budget constraint, fiscal authority (FA) and central bank (CB)

$$P_t G_t + B_{t-1} = M_t - M_{t-1} + \frac{1}{1+i_t} B_t + P_t T_t \tag{17}$$

  "Seignorage" $= \frac{M_t - M_{t-1}}{P_t}$
  Budget constraint in terms of total government liabilities (10)

$$P_t G_t + W_t = \frac{1}{1+i_t} W_{t+1} + \frac{i_t}{1+i_t} M_t + P_t T_t \tag{18}$$

  "Seignorage" $= \frac{i_t}{1+i_t} \frac{M_t}{P_t}$

Split up government in FA and CB
FA budget constraint

$$P_t G_t + B^g_{t-1} = \frac{1}{1+i_t} B^g_t + P_t T_t + P_t z_t \tag{19}$$

B^g_t nominal bonds issued by FA in period t, $P_t z_t$ surplus delivered by CB
CB budget constraint

$$P_t z_t + \frac{1}{1+i_t} B^c_t = M_t - M_{t-1} + B^c_{t-1} \tag{20}$$

B^c_t is quantity of nominal bonds held by CB at end of period t
B_t is quantity of nominal bonds held by households at end of period t

$$B_t = B^g_t - B^c_t$$

Adding (19) and (20) gives (17).
Suppose central bank has no capital: Assets B^c_t = liabilities M_t

$$B^c_t = M_t$$

Use in (20)

$$P_t z_t = \frac{i_t}{1+i_t} M_t$$

"Seignorage" $= \frac{i_t}{1+i_t} \frac{M_t}{P_t}$

- Market equilibrium conditions (in closed economy)
  Goods market equilibrium
  $$C_t + G_t = Y_t \tag{21}$$
  Money market equilibrium already assumed ($M_t = M_t^d = M_t^s$)
  Nominal bond market equilibrium already assumed ($B_t = B_t^d = B_t^s$)
  Real bonds, $b_t = b_t^d = b_t^s = 0$, still price (determine r_t)

- Equilibrium is sequence $\{Y_t, C_t, G_t, T_t, B_t, M_t, P_t, i_t, r_t\}$ fulfilling
  - Budget constraints (period or intertemporal)
    * Private sector
    * Government (consolidated)
  - Private-sector FOCs (pricing equations for r_t, i_t, $1/P_t$)
  - No-Ponzi condition
  - Transversality condition
  - Market equilibrium conditions
    Walras Law, redundancy

- Exogenous and endogenous depends
  - Exogenous: Y_t, G_t, T_t, M_t
  - Endogenous: C_t, B_t, P_t, i_t, r_t
  - Alternatives
  - Optimizing policy
    * Fiscal policy: G_t, T_t endogenous; first-order condition; fiscal objective function, could be "welfare"
    * Monetary policy: M_t endogenous; first-order condition; monetary-policy loss function, could be "welfare"

- Bubbles and self-fulfilling expectations, inflation and deflation
  Assume $G_t = 0, Y_t = Y$ constant
  Assume exogenous constant gross money supply growth

$$M_t = M_0 \mu^t$$

Combine this with (7) to get

$$[u_C(Y, m_t) - u_m(Y, m_t)] m_t = \frac{\beta}{\mu} u_C(Y, m_{t+1}) m_{t+1}$$

where $m_t \equiv M_t / P_t$.
Nonlinear difference equation

$$F(m_t) = \frac{\beta}{\mu} G(m_{t+1})$$

where

$$\begin{aligned} F(m) &\equiv [u_C(Y, m) - u_m(Y, m)] m \\ G(m) &\equiv u_C(Y, m) m \end{aligned}$$

and transversality condition (from (14)) (show!)

$$\lim_{T \to \infty} \beta^T G(m_T) = 0 \tag{22}$$

Forward-looking solution: m_t depends on m_{t+1} which depends on m_{t+2}...

One solution:

$$m_t = m^*$$

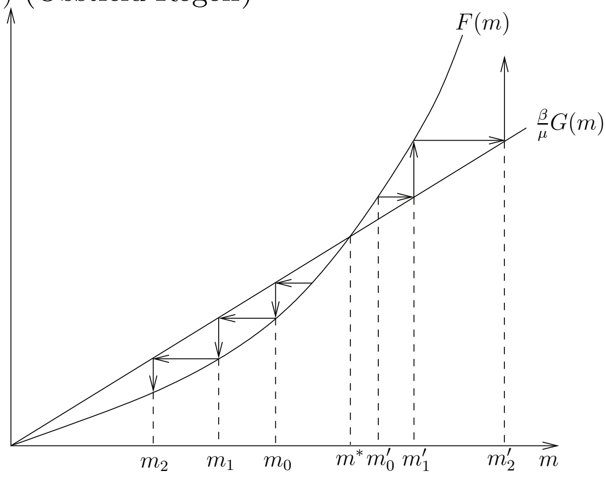
$$\begin{aligned} F(m^*) &= \frac{\beta}{\mu} G(m^*) \\ \frac{u_m(Y, m^*)}{u_C(Y, m^*)} &= 1 - \frac{\beta}{\mu} \end{aligned}$$

$$P_t = \frac{M_t}{m^*} = \frac{M_0}{m^*} \mu^t$$

Exists if $\mu > \beta$ and for $\mu = \beta$ if $u_m(Y, m^*) = 0$, m^* satiation level
Corresponds to non-bubble solution.
Other solutions?
Special case in Obstfeld-Rogoff: $u(C, m) = U(C) + V(m)$; $U(C)$, $V(m)$ concave

- "Hyperdeflation": $m^* < m_0 < m_1 < ...$
  $m_t \to \infty$, $P_t \to 0$
  Not possible if m_t bounded from above, $m_t < \bar{m}$ for some $\bar{m}$ (limited resources (and $B_t \geq 0$!))

– "Hyperinflation": $m^* > m_0 > m_1 > ...$
$m_t \to 0$, $P_t \to \infty$.
Depends on shape of $F(m)$ and $G(m)$ when $m \to 0$. Possible in figure below.
Not possible if central bank "backs" currency and offers to trade it for real goods/assets at some low value $1/\bar{P}$ (high $\bar{P}$) (Obstfeld-Rogoff)



– (How do we know that $F'(m) > \frac{\beta}{\mu}G'(m)$ for $m = m^*$?)

- Two measures of seinorage

$$\begin{aligned}
S_t^1 &\equiv \frac{M_t - M_{t-1}}{P_t} \\
&= \frac{M_t}{P_t} - \frac{M_{t-1}}{P_{t-1}} + \frac{M_{t-1}}{P_{t-1}} - \frac{P_{t-1}}{P_t}\frac{M_{t-1}}{P_{t-1}} \\
&= m_t - m_{t-1} + (1 - \frac{P_{t-1}}{P_t})m_{t-1} \\
&= m_t - m_{t-1} - \frac{1/P_t - 1/P_{t-1}}{1/P_{t-1}}m_{t-1} \\
S_t^2 &\equiv \frac{i_t}{1+i_t}m_t \\
&= \frac{(1+r_t)P_{t+1}/P_t - 1}{(1+r_t)P_{t+1}/P_t}m_t \\
&= (1 - \frac{P_t}{(1+r_t)P_{t+1}})m_t \\
S_t^1 &\not\equiv S_t^2
\end{aligned}$$

Suppose $M_t = M_0\mu^t$, $P_t = P_0\mu^t$, $m_t \equiv \frac{M_t}{P_t} = m$:

$$\begin{aligned} S_t^1 &= (1 - \frac{1}{\mu})m \\ S_t^2 &= (1 - \frac{1}{(1+r)\mu})m \end{aligned}$$

Suppose $\mu = 1$:

$$\begin{aligned} S_t^1 &= 0 \\ S_t^2 &= (1 - \frac{1}{1+r})m > 0 \qquad (r > 0) \end{aligned}$$

- Stochastic states of the world, complete state-contingent markets (Woodford 03-2)
  Household budget constraint in period t

$$P_tC_t + M_t + \tilde{B}_t \leq P_tY_t - P_tT_t + W_t$$

$$\begin{aligned} W_t &= M_{t-1} + A_t \\ \tilde{B}_t &= \mathrm{E}_tQ_{t,t+1}A_{t+1} \end{aligned}$$

W_t beginning-of-period state-dependent nominal value of household's financial wealth
A_t beginning-of-period state-dependent nominal value of household's financial assets other than money
$Q_{t,t+1}$ state-dependent market discount factor ($s^{t+1} = (s_0, s_1, ..., s_{t+1})$, $s_t \in S$), present nominal value in period t of one unit of account in period $t+1$, $Q_{t,t} = 1$
$\tilde{B}_t$ end-of-period value of household's financial assets other than money
Present value in period t of one riskless unit of account in period $t+1$

$$\frac{1}{1+i_t} = \mathrm{E}_tQ_{t,t+1}$$

$$\mathrm{E}_tQ_{t,t+1}W_{t+1} = \mathrm{E}_tQ_{t,t+1}(M_t + A_{t+1}) = \frac{1}{1+i_t}M_t + \tilde{B}_t$$

Rewrite budget constraint

$$P_tC_t + \frac{i_t}{1+i_t}M_t + \mathrm{E}_tQ_{t,t+1}W_{t+1} \leq P_tY_t - P_tT_t + W_t$$

No-Ponzi condition

$$W_{t+1} \geq -\mathrm{E}_{t+1}\sum_{s=t+1}^{\infty} Q_{t+1,s}(P_sY_s - P_sT_s)$$

$$\mathrm{E}_{t+1}\sum_{s=t+1}^{\infty} Q_{t+1,s}(P_sY_s - P_sT_s) < \infty$$

where $Q_{t,T} = \prod_{s=t}^{T-1} Q_{s,s+1}$.

Transversality condition

$$\lim_{T\to\infty} \mathrm{E}_tQ_{t,T}W_T = 0$$

Intertemporal budget constraint

$$\mathrm{E}_t\sum_{s=t}^{\infty} Q_{t,s}\left(P_sC_s + \frac{i_s}{1+i_s}M_s\right) = \mathrm{E}_t\sum_{s=t}^{\infty} Q_{t,s}(P_sY_s - P_sT_s) + W_t$$

Preferences (ξ_s exogenous random preference shock in period s)

$$U_t = \mathrm{E}_t\sum_{s=t}^{\infty}\beta^{s-t}\left[u\left(C_s, \frac{M_s}{P_s}, \xi_s\right) + v(G_s, \xi_s)\right]$$

Discount factor

$$Q_{t,t+1} = \frac{\beta u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}\right)/P_{t+1}}{u_C\left(C_t, \frac{M_t}{P_t}, \xi_t\right)/P_t}$$

– Substitute for C_t, take FOCs for W_{t+1}

– Lagrangian, FOCs

$$\begin{aligned} U_{Ct} &= \lambda P_t \\ U_{C,t+1} &= \lambda P_{t+1}Q_{t,t+1} \end{aligned}$$

Real price of real bond

$$\frac{1}{1+r_t} = \mathrm{E}_t Q_{t,t+1} P_{t+1}/P_t = \frac{\beta \mathrm{E}_t u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}\right)}{u_C\left(C_{t,} \frac{M_t}{P_t}, \xi_t\right)} \tag{23}$$

Nominal price of nominal bond

$$\frac{1}{1+i_t} = \mathrm{E}_t Q_{t,t+1} = \frac{\beta \mathrm{E}_t \left[u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}\right)/P_{t+1}\right]}{u_C\left(C_{t,} \frac{M_t}{P_t}, \xi_t\right)/P_t} \tag{24}$$

Real value of money

$$\frac{1}{P_t} = \frac{u_{M/P}\left(C_t, \frac{M_t}{P_t}, \xi_t\right)/P_t}{u_C\left(C_t, \frac{M_t}{P_t}, \xi_t\right)} + \frac{\beta \mathrm{E}_t \left[u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}\right)/P_{t+1}\right]}{u_C\left(C_t, \frac{M_t}{P_t}, \xi_t\right)} \tag{25}$$

$$\frac{1}{P_t} = \frac{\beta \mathrm{E}_t \left[u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}\right)/P_{t+1}\right]}{u_C\left(C_t, \frac{M_t}{P_t}, \xi_t\right) - u_{M/P}\left(C_t, \frac{M_t}{P_t}, \xi_t\right)} \tag{26}$$

Fisher equation under uncertainty, combine (23) and (24), (8) no longer exact (inflation risk premium, covariance, Jensen's inequality).

$$\frac{1+i_t}{1+r_t} = \frac{\mathrm{E}_t u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}\right)}{\mathrm{E}_t \left[u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}\right) P_t/P_{t+1}\right]}$$

Equation (9) holds under uncertainty

- The Fisher equation under uncertainty

$$\begin{aligned}
\ln(1+i_t) &= -\ln \frac{\beta \mathrm{E}_t \left[u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}\right)/P_{t+1}\right]}{u_C\left(C_{t,} \frac{M_t}{P_t}, \xi_t\right)/P_t} = -\ln \mathrm{E}_t q_{t,t+1}/\frac{P_{t+1}}{P_t} \\
\ln(1+r_t) &= -\ln \frac{\beta \mathrm{E}_t u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}\right)}{u_C\left(C_{t,} \frac{M_t}{P_t}, \xi_t\right)} = -\ln \mathrm{E}_t q_{t,t+1}
\end{aligned}$$

Real discount factor $q_{t,t+1} \equiv \frac{\beta u_C\left(C_{t+1}, \frac{M_{t+1}}{P_{t+1}}, \xi_{t+1}\right)}{u_C\left(C_{t,} \frac{M_t}{P_t}, \xi_t\right)}$. Assume $q_{t,t+1}$ and $\frac{P_{t+1}}{P_t}$ jointly lognormal ($u_C(C_{t,} \frac{M_t}{P_t}, \xi_t$ and P_t known in period t) and use the results:

– For x normal (show!),

$$\mathrm{E}[\exp x] = \exp(\mathrm{E}[x] + \frac{1}{2}\mathrm{Var}[x]).$$

It follows that for x and y jointly lognormal,

$$\begin{aligned}
\mathrm{E}[x] &= \exp\left(\mathrm{E}[\ln x] + \frac{1}{2}\mathrm{Var}[\ln x]\right) \\
\mathrm{E}[xy] &= \mathrm{E}[x]\mathrm{E}[y]\exp\left(\mathrm{Cov}[\ln x, \ln y]\right)
\end{aligned}$$

$$
\begin{aligned}
\pi_{t+1} &\equiv \ln \frac{P_{t+1}}{P_t} \equiv p_{t+1} - p_t \\
\ln(1+r_t) &= -\ln \mathrm{E}_t q_{t,t+1} \\
\ln(1+i_t) &= -\ln \mathrm{E}_t \left[q_{t,t+1} \frac{P_t}{P_{t+1}} \right] \\
&= -\ln \left(\mathrm{E}_t[q_{t,t+1}] \mathrm{E}_t \left[\frac{P_t}{P_{t+1}} \right] \exp\{\mathrm{Cov}_t[\ln q_{t,t+1}, -\pi_{t+1}]\} \right) \\
&= -\ln \mathrm{E}_t q_{t,t+1} - \ln \mathrm{E}_t \left[\frac{P_t}{P_{t+1}} \right] + \mathrm{Cov}_t[\ln q_{t,t+1}, \pi_{t+1}] \\
&= -\ln \mathrm{E}_t q_{t,t+1} - \ln \exp \left(\mathrm{E}_t \left[\ln \frac{P_t}{P_{t+1}} \right] + \frac{1}{2} \mathrm{Var}_t \left[\ln \frac{P_t}{P_{t+1}} \right] \right) \\
&\quad + \mathrm{Cov}_t[\ln q_{t,t+1}, \pi_{t+1}] \\
&= \ln(1+r_t) + \mathrm{E}_t \pi_{t+1} - \frac{1}{2} \mathrm{Var}_t [\pi_{t+1}] + \mathrm{Cov}_t[\ln q_{t,t+1}, \pi_{t+1}] \qquad (27)
\end{aligned}
$$

Inflation risk premium

$$-\frac{1}{2} \mathrm{Var}_t [\pi_{t+1}] + \mathrm{Cov}_t[\ln q_{t,t+1}, \pi_{t+1}]$$

(see Svensson 93 for more on term, inflation, and foreign exchange risk premia)
Disregard inflation risk premium, approximate Fisher equation

$$\ln(1+i_t) = \ln(1+r_t) + \mathrm{E}_t \pi_{t+1}$$

- Relation between stochastic nominal discount factors $Q_{t,\tau}$ and prices of Arrow-Debreu history-contingent contracts

  Stochastic events $s_\tau \in S$ in period $\tau \geq t$, finite for simplicity

  History $s_t^\tau = (s_t, s_{t+1}, ..., s_\tau) \in S^{\tau-t+1}$ up to and including period $\tau \geq t$

  $\pi_{t,\tau}(s_t^\tau)$ probability of s_t^τ conditional on s_t

  Consumption C_τ of one consumption good in period τ, real balances m_τ, subjective discount factor β $(0 < \beta < 1)$

  Expected utility ($G_t \equiv 0$)

$$\mathrm{E}\left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} u(C_\tau, m_\tau, s_\tau)] | s_t \right] = \sum_{\tau=t}^{\infty} \sum_{s_t^\tau \in S^\tau} \beta^{\tau-t} u[C_\tau(s_t^\tau), m_\tau(s_t^\tau), s_\tau] \pi_{t,\tau}(s_t^\tau)$$

  Assume complete markets

  $\tilde{p}_{t,\tau}(s_t^\tau)$ Arrow-Debreu price, price of delivery of one consumption good in history s_t^τ in period τ in terms of consumption goods in period t and event s_t

  In equilibrium

$$\tilde{p}_{t,\tau}(s_t^\tau) = \frac{\beta^{\tau-t} u_C[C_\tau(s_t^\tau), m_\tau(s_t^\tau), s_\tau] \pi_{t,\tau}(s_t^\tau)}{u_C[C_t(s_t), m_t(s_t), s_t]}$$

  $Q_{t,\tau}(s_t^\tau)$ history-contingent nominal discount factor

  In equilibrium

$$Q_{t,\tau}(s_t^\tau) = \frac{\beta^{\tau-t} u_C[C_\tau(s_t^\tau), m_\tau(s_t^\tau), s_\tau]/P_\tau(s_t^\tau)}{u_C[C_t(s_t), m_t(s_t), s_t]/P_t(s_t)}$$

  (where $P_\tau(s_t^\tau)$ is the nominal price of consumption goods in period τ)

Thus, the Arrow-Debreu prices and the nominal discount factors are related according to

$$\begin{aligned}\tilde{p}_{t,\tau}(s_t^\tau) &= Q_{t,\tau}(s_t^\tau)\frac{P_\tau(s_t^\tau)}{P_t(s_t)}\pi_{t,\tau}(s_t^\tau), \\ \tilde{p}_{t,t}(s_t) &= 1.\end{aligned}$$

It follows that the intertemporal budget constraint in terms of E_t and $Q_{t,\tau}$,

$$\mathrm{E}_t \sum_{\tau=t}^{\infty} Q_{t,\tau} P_\tau \left(C_\tau + \frac{i_\tau}{1+i_\tau} m_\tau \right) = \mathrm{E}_t \sum_{\tau=t}^{\infty} Q_{t,\tau} P_\tau (Y_\tau - T_\tau) + W_t,$$

is equivalent to the Arrow-Debreu budget constraint,

$$\begin{aligned}&\sum_{\tau=t}^{\infty} \sum_{s_t^\tau \in S^\tau} \tilde{p}_{t,\tau}(s_t^\tau) \left(C_\tau(s_t^\tau) + \frac{i_\tau}{1+i_\tau}(s_t^\tau) m_\tau(s_t^\tau) \right) \\ &= \sum_{\tau=t}^{\infty} \sum_{s_t^\tau \in S^\tau} \tilde{p}_{t,\tau}(s_t^\tau)[Y_\tau(s_t^\tau) - T_\tau(s_t^\tau)] + \frac{W_t}{P_t(s_t)}.\end{aligned}$$

AD market structure: All markets in period 0. No bonds, etc. (redundant) (money?).

Friedman's rule Socially optimal real balances, "the optimum quantity of money"

Supply money at zero real cost (approximately)

$$\max_{\frac{M_t}{P_t}} u(C_t, \frac{M_t}{P_t})$$

Assume there exists a satiation level for real balances (may depend on C_t)

$$U_{M/P}(C_t, \frac{M_t}{P_t}) = 0$$

From (9), optimum quantity of money

$$\frac{M_t}{P_t} = \min L(C_t, 0)$$

Friedman's 69 rule,

$$i_t \equiv 0$$

Consequences

Under perfect foresight, from (8), $i_t = 0 \Rightarrow$

$$\frac{P_{t+1}}{P_t} = \frac{1}{1+r_t}$$

($r_t > 0 \Rightarrow$ deflation)

Under uncertainty, from (27), $i_t = 0 \Rightarrow$

$$\mathrm{E}_t \pi_{t+1} = -\ln(1+r_t) + \frac{1}{2}\mathrm{Var}_t\,[\pi_{t+1}] - \mathrm{Cov}_t[\ln q_{t,t+1}, \pi_{t+1}]$$

- Seignorage, Phelps: optimal inflation tax (Walsh 03 sect. 4.4)
- Welfare gain of going to the Friedman rule (Lucas 00, problem in extending money-demand function beyond observations)
- Cf. deflation and liquidity trap in Japan (to be discussed later). Role for monetary policy in stabilizing economy?