New Techniques to Extract Market Expectations from Financial Instruments

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1 Introduction

Central banks have several reasons for extracting information from asset prices. Asset prices may embody more accurate and more up-to-date macroeconomic data than what is currently published or directly available to policy makers. Aberrations in some asset prices may indicate imperfections or manipulations relevant for banking and financial market surveillance. Especially, asset prices will reflect market participants' expectations about the future, which is the focus of this paper.

In fixed exchange rate regimes, central banks have an obvious interest in assessing the credibility of the regime and the likelihood of future speculative attacks. For this purpose, differentials between domestic and foreign interest rates, forward exchange rates, and prices on exchange rates and option prices are used to estimate expectations of realignments and other regime changes. With floating exchange rates and various forms of inflation targeting, central banks also have an obvious interest in assessing the credibility of the regime and market expectations of future monetary policy.

This paper is a selective survey of new or recent methods to extract information about market expectations from asset prices for monetary policy purposes. We shall discuss methods to extract

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market expectations of future interest rates, exchange rates and inflation rates. Traditionally, interest rates and forward exchange rates have been used to extract expected means of future interest rates, exchange rates and inflation. More recently, these methods have been refined to rely on implied forward interest rates, so as to extract expected future time-paths of interest rates, exchange rates and inflation rates. Very recently, methods have been designed to extract not only means but the whole (risk neutral) probability distribution of future interest rates and exchange rates from a set of option prices.

More developed and deeper financial markets, increased international financial integration, and new financial instruments are preconditions for these methods. The survey also reports on available instruments and their suitability for the different purposes and methods.

Section 2 of the paper discusses extraction of interest rate expectations. This section covers some basic theory of bond pricing, available financial instruments and their suitability, estimation of forward rates, various risk premia, and estimation of the whole risk-neutral probability distribution. Section 3 and 4 apply the results in section 2 to discuss estimation of exchange rate expectations and inflation expectations, respectively. Section 5 concludes. An appendix contains definitions and technical details.

2 Interest rate expectations

2.1 Interest rates, forward rates and term premia

Let $B(t,T)$ denote the price in units of account at time t (the *trade date*) of a discount bond (zero-coupon bond) paying one unit of account at time T (the *maturity date*), $T > t$, where time is measured in years. We will call $T - t$ the *maturity* of the bond. Let $i(t,T)$ denote the corresponding continuously compounded (spot) *interest rate* on the discount bond. It is defined by

$$i(t,T) = -\frac{b(t,T)}{T-t}, \tag{2.1}$$

where $b(t,T)$ denotes the log of $B(t,T)$. A (spot) *yield curve* is a plot of interest rates $i(t,T)$ as a function of different maturity dates T, for a given trade date t. It is one of several ways to represent the term structure of interest rates.

Let $f(t,\tau,T)$ denote the continuously compounded *forward (interest) rate* at trade date t, for a *forward contract* with *settlement date* τ and maturity date T, $t \leq \tau < T$. The contract stipulates buying at time τ, at the price $B(t,\tau,T) = e^{-f(t,\tau,T)(T-\tau)}$, a discount bond that pays one unit of account at time T. Absence of arbitrage (see Appendix) requires that the forward interest rate fulfills

$$f(t,\tau,T) = \frac{i(t,T)(T-t) - i(t,\tau)(\tau-t)}{T-\tau}. \tag{2.2}$$

Alternatively, if no explicit forward market exists, (2.2) can be interpreted as the definition of an *implied* forward rate, a forward rate that is implied by the term structure of interest rates $i(t,T)$.

The instantaneous forward rate, $f(t,\tau)$, is defined as the limit

$$f(t,\tau) = \lim_{T\to\tau} f(t,\tau,T) \tag{2.3}$$

and refers to (hypothetical) forward contracts of infinitesimal maturity.

When expressed as continuously compounded rates, forward interest rates and spot interest rates are conveniently related exactly as marginal and average cost (where $\tau - t$ corresponds to the quantity), and they fulfill the corresponding relations

$$f(t,t+m) = i(t,t+m) + m\frac{\partial i(t,t+m)}{\partial m} \tag{2.4}$$

$$i(t,t+m) = \frac{1}{m}\int_{s=0}^{m} f(t,t+s)ds. \tag{2.5}$$

A *forward rate curve* is a plot of forward rates $f(t,\tau,\tau+m)$ as a function of different settlements dates τ, for a given trade date t and a given maturity m. It is an alternative way of representing the term structure of interest rates, which is often convenient for monetary policy purposes.

In a monetary policy context, forward interest rates are potentially useful as indicators of market expectations of future interest rates. Let us assume that market participants have rational expectations (we will consider the possibility of non-rational expectations in section 2.3). The forward rate, $f(t,\tau,T)$, differs from the future interest rate expected by market participants, $\mathrm{E}_t i(\tau,T)$, by the *forward term premium*, $\varphi^f(t,\tau,T)$. With either assumptions or information about the forward term premium, expected future interest rates can then be inferred from forward rates,

$$\mathrm{E}_t i(\tau,T) = f(t,\tau,T) - \varphi^f(t,\tau,T). \tag{2.6}$$

Theoretical expressions for the spot and forward interest rates and the term premium can be derived under the assumption that bonds are priced with a *stochastic discount factor* (SDF) (pricing kernel, state prices), $D(t,T)$.[1] Such pricing implies that an asset with a stochastic

[1] Any bond pricing theory that assumes no arbitrage, or a general equilibrium setting where no arbitrage opportunities remain unexploited, imply that a stochastic discount factor exists (see for instance Duffie [29]).

payoff $x(T)$ at time T has a price $V(t,T)$ at date $t < T$ given by $V(t,T) = \mathrm{E}_t[D(t,T)x(T)]$. It follows that a discount bond that matures at time T with a fixed payoff of one unit of account has the price

$$B(t,T) = \mathrm{E}_t D(t,T). \tag{2.7}$$

Assume further that $D(t,T)$, conditional upon information available at the trade date t, is *lognormally* distributed. Then explicit expressions for the interest rate, the forward rate and the forward premium can be derived (see Appendix).

The forward term premium can be expressed in several different ways, one of which is

$$\varphi^f(t,\tau,T) = -\frac{1}{2}(T-\tau)\mathrm{Var}_t i(\tau,T) - \mathrm{Cov}_t[d(t,\tau), f(t,\tau,T) - i(\tau,T)], \tag{2.8}$$

where $d(t,T) = \ln D(t,T)$. The first term on the right-hand side is a Jensen inequality term. The second term is the familiar covariance between the SDF and an excess return, $f(t,\tau,T) - i(\tau,T)$, the excess return on a forward investment relative to a spot investment at date τ. The higher the covariance, the more attractive a forward investment, and the lower the forward term premium. The forward term premium is frequently assumed to be negligible, or at least constant; we will look into this in section 2.4.

The precise form of the SDF follows from what specific asset pricing theory is assumed. For instance, consider consumption-based asset pricing with an additively separable utility function with constant relative risk aversion. For *nominal* interest rates, the log (nominal) discount factor is

$$d(t,T) = -\gamma[c(T) - c(t)] - [p(T) - p(t)] - \rho(T-t), \tag{2.9}$$

where $\gamma \geq 0$ and ρ denote the degree of relative risk aversion and the rate of time preference, respectively, $c(t)$ denotes log real consumption and $p(t)$ denotes the log of a consumer price index. For *real* interest rates, the log (real) discount factor is like (2.9) but without the price terms.[2]

A given stochastic process for consumption (and for the price level, if we consider nominal interest rates) will result in a stochastic process for $d(t,T)$ and hence for $i(t,T)$, $f(t,\tau,T)$ and $\varphi^f(t,\tau,T)$. For instance, if inflation is an AR(1) process, and consumption growth is white noise, then the log nominal SDF is an ARMA(1,1). Backus, Foresi, and Zin [5] show that this gives a discrete time version of Vasicek's [91] model for the short interest rate and the yield curve.

[2] Utility functions incorporating habit formation or relative consumption considerations will give alternative SDFs.

2.2 Instruments and forward rates

Which financial instruments are available for extracting interest rate expectations? We can here distinguish between explicit forward rates, where the market quotes are forward rates, and other financial instruments that allow implicit forward rates to be extracted.

Explicit forward rates can be observed on forward contracts on deposits, bills and bonds. So called forward rate agreements, FRA's, are especially relevant. They are forward rates on Euro deposit and exist for a number of currencies, but as far as we know only for settlement and maturities up to a year. For the US, federal funds futures rates are closely watched to assess markets' judgements of near-term policy (cf. Rudebusch [76]).

Since explicit forward rates only exist for a limited range of times to settlement and times to maturity, it is very common to use other interest rate bearing assets to infer implicit forward rates. Discount bonds, for instance so called strips on government bonds, make it straightforward to extract forward rates for the grid of maturities available, using equation (2.2) above. For short maturities (up to one year), discount bonds exist in the form of bills and deposits, whereas for longer maturities (above one year), coupon bonds are much more frequent than discount bonds, making the estimation of forward rates more involved.

Ideally one should use instruments with high liquidity, with insignificant credit risk, and without distorting tax treatment. In practice, a high degree of institutional knowledge is required in order to select suitable instruments. The desirability of low credit risk points (in most countries) towards using treasury bills and government bonds, if they are sufficiently liquid and not subject to distorting tax treatment. Using government bonds and bills have indeed become quite common in forward rate estimation. However, in some countries, there are few and illiquid treasury bills, or different kinds of government bonds with very different liquidity or subject to very different tax treatment.

Cross-country differences for treasury bills and government bonds make cross-currency comparisons more difficult. A possible alternative that provide more standardization across currencies is to rely on Euro deposit (LIBOR) rates for shorter maturities and interest swap rates for longer maturities.[3] These markets are very liquid, and contracts, including tax treatment, are standardized across currencies. One drawback is the credit risk, since these contracts are supplied by private banks, but for a given major bank the credit risk should be similar across cur-

[3] Swap rate quotes can be interpreted as par bond rates, that is, rates on bonds for which the coupon rate and the yield to maturity are equal.

rencies and maturities and (hopefully) rather small and stable. Comparative studies of implicit forward rates estimated from different data sources would be desirable.

2.2.1 Estimation of implicit forward rates

Estimation of implicit forward rates can be based on a structural model for interest rate dynamics, or be simple curve fitting. The former is relevant if the purpose is to predict future changes in the yield curve. The latter is relevant when the purpose is to extract interest rate expectations without necessarily making the extraction conditional upon a particular model. Here we only discuss curve-fitting (Andersen et al. [1] on models of interest rates).[4]

There are several different curve-fitting methods to estimate implicit forward rates (see Svensson [87] for references, and Andersen et al. [1] for a detailed survey and a comparison between the methods). Most methods follow McCulloch [62] and [63] in fitting theoretical discount bond prices (a discount function) so as to explain observed bond prices or yields to maturity for a particular trade date. For monetary policy purposes relatively parsimonious functional forms give sufficient precision. In Svensson [87] and [89] a parsimonious yet flexible form for the forward rate function, an extension of that of Nelson and Siegel [69], has been suggested. It is now used regularly by several central banks.

Let $f(m)$ denote the instantaneous forward rate $f(t, t+m)$ with time to settlement m, for a given trade date t. Then the *extended Nelson and Siegel forward rate function* is

$$f(m;b) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right), \qquad (2.10)$$

where $\beta = (\ \beta_0,\ \beta_1,\ \beta_2, \tau_1, \beta_3, \tau_2)$ is a vector of parameters (β_0, τ_1 and τ_2 must be positive).

The forward rate in (2.10) consists of four components. The first component is a constant, β_0. This the horizontal asymptote of the function, and it must be positive to ensure that long forward rates are positive. The second component, the exponential term $\beta_1 \exp\left(-\frac{m}{\tau_1}\right)$, is monotonically decreasing (or increasing, if β_1 is negative) towards zero as a function of the time to settlement. When the time to settlement approaches zero, the forward rate approaches the constant $\beta_0 + \beta_1$, which must be non-negative to ensure that the instantaneous (in practice overnight) interest rate is non-negative. The third component generates a hump-shape (or a U-shape, if β_2 is negative) as a function of the time to settlement, $\beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right)$. The fourth is another hump- or U-shape. This function hence allows up to two hump- or U-shapes. The

[4] Estimation of implicit real forward interest rates is discussed in section 4.

original Nelson and Siegel function has only one hump- or U-shape.[5]

Let $i(m)$ denote the spot interest rate $i(t, t+m)$ with time to maturity m, for a given trade date t. The interest rate function, $i(m; \beta)$, that corresponds to the above functional form is easily derived by integrating (2.10) according to (2.5). Let $B(m; \beta)$ denote the corresponding price of a discount bond $B(t, t+m)$, called the *discount function* in the literature,

$$B(m; \beta) = \exp\left[-i(m; \beta)m\right]. \tag{2.11}$$

The discount function is estimated for each trade date by minimizing either (the sum of squared) price errors or (the sum of squared) yield errors (see Appendix for details). In the former case, the discount function is used to compute estimated bond prices for given parameters. The parameters are then chosen so as to minimize the sum of squared errors between the estimated and observed bond prices. This is the standard way since McCulloch [62] and [63].

Minimizing price errors sometimes results in fairly large yield errors for bonds and bills with short maturities, since prices are very insensitive to yields for short maturities.[6] It may be better to choose the parameters so as to minimize yield errors, in particular since the focus in monetary policy analysis is on interest rates rather than prices. Then the estimated yield to maturity for each bond is computed for given parameters and the parameters are chosen so as to minimize the sum of squared yield errors between estimated yields and observed yields.

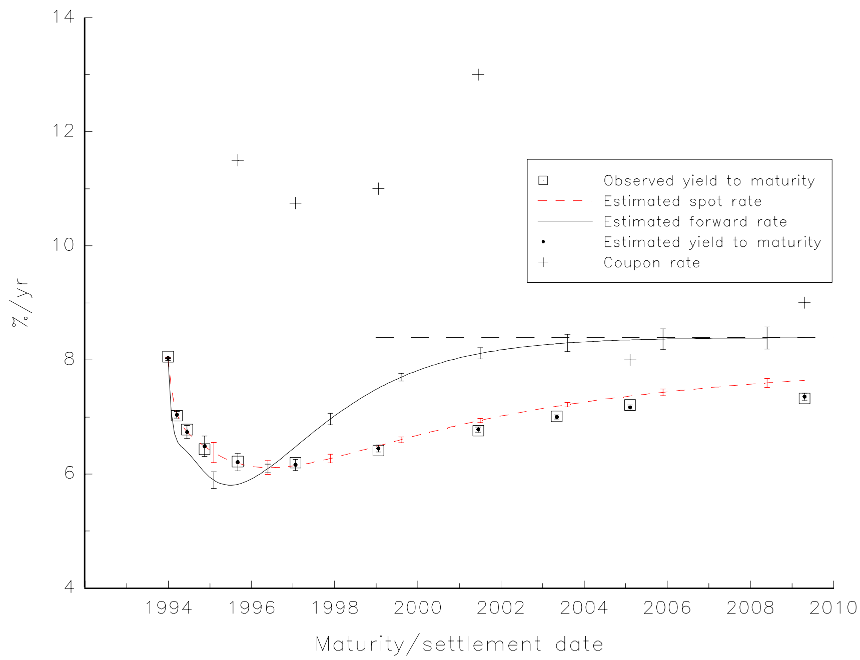
In many cases the original Nelson and Siegel model gives a satisfactory fit. In some cases when the term structure is more complex, the extended model may improve the fit considerably.

As an example of an estimation result, see *Figure 2.1.* It shows the estimate for Sweden for the trade date December 29, 1993. The estimation is done with minimized yield errors, for the extended Nelson and Siegel model. The squares show the observed marginal lending rate and observed yields to maturity on Treasury bills and Government benchmark bonds, plotted against the maturity date. The pluses show the coupon rates for the bonds (the Treasury bills and the marginal lending rate have zero coupons). The dashed curve shows the estimated spot rate curve, the zero-coupon rates. The error bars show 95 percent confidence intervals (computed by the delta method, see Appendix). Dots with error bars show the estimated yields to maturity with 95 percent confidence intervals (the estimated yields are on the spot rate curve for zero-coupon bonds but are generally not so for coupon bonds, since yields to maturity on coupon

[5] The assumption of a horizontal asymptote implies that the forward premium in (2.8) approaches a constant, which in turn requires that $\mathrm{Var}_t i(\tau, T)$ approaches zero.

[6] Recall that the elasticity of the price with respect to the continuously compounded yield (with respect to one plus the annual effective yield) is equal to (the negative of) the duration of a bond (the present-value weighted average maturity of coupon payments and face value).

Figure 2.1: Implicit spot and forward rates, Sweden, Dec 29, 1993, minimized yield errors, extended N&S, 95% confidence interval



bonds generally differ from yields to maturity on zero-coupon bonds). The fit is good, and the error bars are hardly visible. The solid curve shows the estimated (instantaneous) forward rate, plotted against the settlement date, with error bars showing 95 percent confidence intervals. The horizontal dashed line is the asymptote for the spot and forward rate (the parameter β_0). The root mean squared yield error for this estimation is 3 basis percentage points per year, the root mean squared price error is 0.16 kronor for a bond with face value 100 kronor.

The forward rate has a somewhat complex shape on December 29, 1993, with a conspicuous kink for about 3 months settlement. Therefore, the fit with the original Nelson and Siegel functional form is unsatisfactory, and the extended variant results in a much better fit.[7]

2.3 Extracting risk-neutral distributions from bond options

A European (bond) call option gives the holder the right, but not the obligation, to buy a bond (with maturity date T) for the strike price, X, at the expiry date, $\tau < T$. The call option price

[7] A possible explanation for the kink is speculation in the media and in financial markets that Sveriges Riksbank under the new Governor, Urban Bäckström, whose tenure was to begin on January 1, 1994, would embark upon more expansionary monetary policy.

at the trade date $t < \tau$, $C(t, \tau; X)$, satisfies

$$C(t, \tau; X) = \mathrm{E}_t \{D(t, \tau) \max[0, B(\tau, T) - X]\}, \tag{2.12}$$

where $D(t, \tau)$ is the nominal discount factor and $B(\tau, T)$ is the price at date τ of the bond. A standard result in option pricing theory (see Cox and Ross [24]) is that the call price can be thought of as a discounted "risk-neutral(-ized)" expected value (denoted by $\widetilde{\mathrm{E}}_t$) of the payoff,

$$C(X) = e^{-i(t,\tau)(\tau-t)} \widetilde{\mathrm{E}}_t \max[0, B(\tau, T) - X], \tag{2.13}$$

where we have dropped the date arguments in the call price. Compared with (2.12), the risk-neutral part of the nominal discount factor, $D(t, \tau)$, is loaded into the interest rate term, and the remaining part accounts for the difference between E_t and $\widetilde{\mathrm{E}}_t$, as we will see below.

In terms of the log bond price, $b = \ln B(\tau, T)$, (2.13) can also be written as

$$C(X) = e^{-i(t,\tau)(\tau-t)} \int_{\ln X}^{\infty} \left(e^b - X\right) \widetilde{h}(b)\, db, \tag{2.14}$$

where $\widetilde{h}(b)$ is the risk-neutral (univariate) probability density function. Differentiating (2.14) with respect to the strike price and rearranging gives the risk-neutral distribution function

$$\widetilde{\Pr}(B \leq X) = 1 + e^{i(t,\tau)(\tau-t)} \frac{\partial C(X)}{\partial X}. \tag{2.15}$$

Differentiating once more, and changing the variable to b, gives the risk-neutral probability density function of b,

$$\widetilde{h}(b) = e^{[i(t,\tau)(\tau-t)+b]} \frac{\partial^2 C(e^b)}{\partial X^2}. \tag{2.16}$$

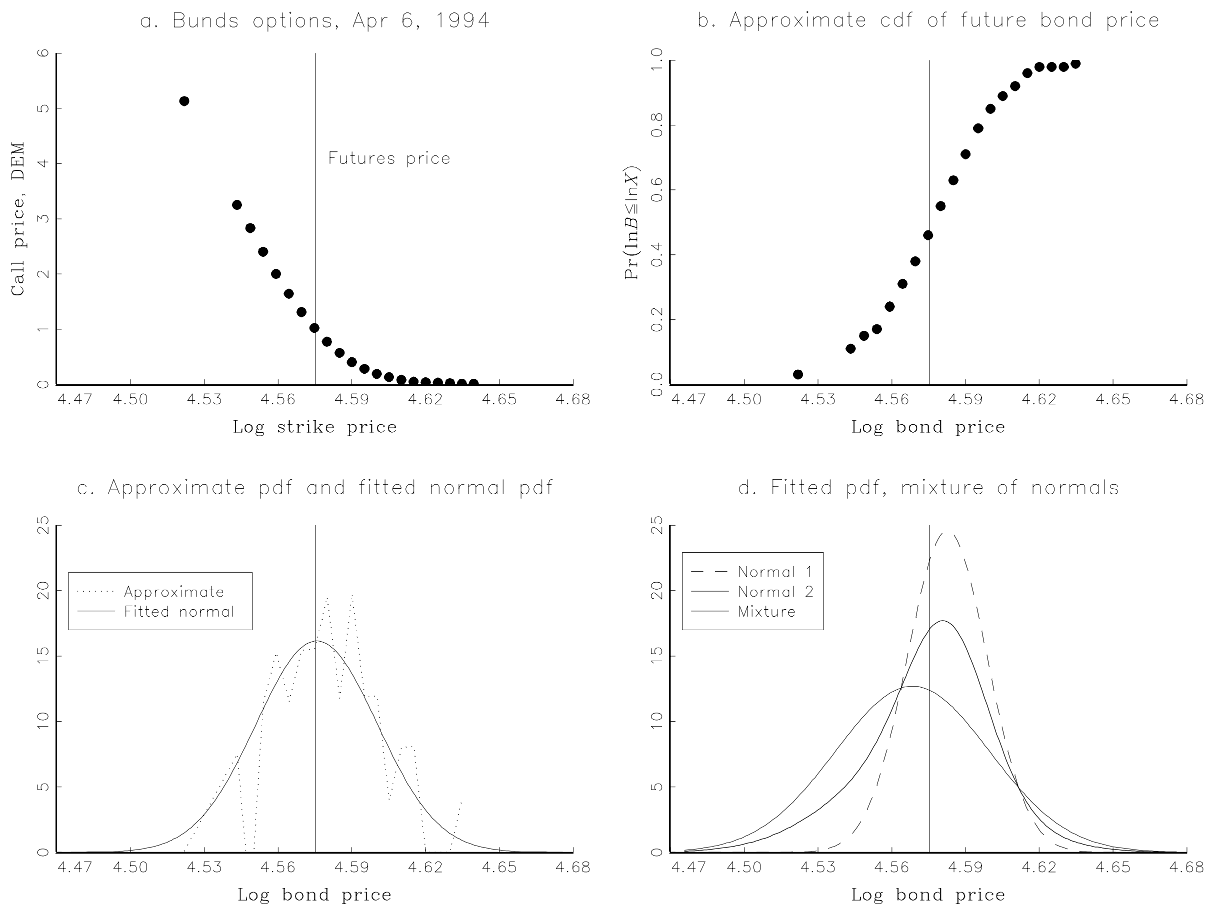
To illustrate this, we show in *Figure 2.2.a*, for the trade date April 6, 1994, prices of LIFFE German Bund options, expiring in June 1994, for strike prices $\{X_1, ..., X_K\}$.[8] The futures price, marked with a vertical line, can be treated as an undiscounted option with a zero strike price. A difference quotient approximation of the derivative in (2.15) gives the approximate distribution function in *Figure 2.2.b*.[9] The approximate probability density function, obtained by a second-order difference quotient, is shown as the dotted curve in *Figure 2.2.c*.[10] The

[8] The options are written on Bund (approximately 10-year German Government Bonds with an annual 6% coupon) futures. The DEM futures and options prices are for a bond with a face value of DEM 100. The futures-style margining on the exchange LIFFE implies that the options are de facto European, and that the call price is paid at the expiration date. This gives the same call pricing formula as (2.18), but without discounting (set $i = 0$). It also implies that price quotes for options without trade but with open positions contain useful information. See Chen and Scott [22] for details.

[9] This approach is suggested by Neuhaus [70]. We use $\frac{\partial C_t}{\partial X} \approx \frac{1}{2}\left[\frac{C(X_{i+1}) - C(X_i)}{X_{i+1} - X_i} + \frac{C(X_i) - C(X_{i-1})}{X_i - X_{i-1}}\right]$.

[10] This approach was first suggested by Breeden and Litzenberger [14]. We use $\frac{\partial^2 C_t}{\partial X^2} \approx \left[\frac{C(X_{i+1}) - C(X_i)}{X_{i+1} - X_i} - \frac{C(X_i) - C(X_{i-1})}{X_i - X_{i-1}}\right] / \left[\frac{1}{2}(X_{i+1} - X_{i-1})\right]$.

Figure 2.2: Data and fitted distributions for Bund options April 6, 1994.



approximate distribution function is decreasing in some intervals, and the approximate density function has some negative values and is very jagged. This could possibly be explained by some abberations of the option prices, but more likely by the approximation of the derivatives: changing approximation method (for instance, from centred to forward difference quotient) can have a strong effect on the results, but all methods generate strange results in some interval.

It is therefore advantageous to impose restrictions which guarantee that the result is a reasonable probability distribution. Ritchey [72], Melick and Thomas [64] and Bahra [6] assume that the risk-neutral distribution is a mixture of univariate log-normals. We show that this can be generated by a true distribution of the log stochastic discount factor, d, and the log bond price, b, which is a mixture of *bi*variate normals. Let $\phi\left(x;\mu^j,\Omega^j\right)$ denote a normal multivariate pdf over x with mean vector μ^j and covariance matrix Ω^j, and let α^j be the weight of the j^{th}

pdf. The true bivariate probability density function is assumed to be

$$\text{pdf}\left(\begin{bmatrix} d \\ b \end{bmatrix}\right) = \sum_{j=1}^{n} \alpha^j \phi\left(\begin{bmatrix} d \\ b \end{bmatrix}; \begin{bmatrix} \bar{d}^j \\ \bar{b}^j \end{bmatrix}, \begin{bmatrix} \sigma_{dd}^j & \sigma_{db}^j \\ \sigma_{db}^j & \sigma_{bb}^j \end{bmatrix}\right), \text{ with } \sum_{j=1}^{n} \alpha^j = 1 \text{ and } \alpha^j \geq 0. \quad (2.17)$$

For $n = 1$, this is the same type of distribution as used in Section 2.1.[11]

It is shown in the Appendix that (2.17), together with the assumption that $\bar{d}^j = \bar{d}$ and $\sigma_{dd}^j = \sigma_{dd}$ imply that the call price in (2.12) is

$$\begin{aligned} C(X) &= e^{-i(t,\tau)(\tau-t)} \sum_{j=1}^{n} \alpha^j \left[\exp\left(\bar{b}^j + \frac{1}{2}\sigma_{bb}^j + \sigma_{db}^j\right) \Phi\left(\frac{\bar{b}^j + \sigma_{bb}^j + \sigma_{db}^j - \ln X}{\sqrt{\sigma_{bb}^j}}\right)\right. \\ &\quad \left. - X\Phi\left(\frac{\bar{b}^j + \sigma_{db}^j - \ln X}{\sqrt{\sigma_{bb}^j}}\right)\right], \end{aligned} \quad (2.18)$$

where $\Phi\,(\;)$ denotes the standardized normal distribution function. Similarly, the forward price of the bond is

$$B(t, \tau, T) \equiv e^{-f(t,\tau,T)(T-\tau)} = \sum_{j=1}^{n} \alpha^j \exp\left(\bar{b}^j + \sigma_{db}^j + \frac{\sigma_{bb}^j}{2}\right). \quad (2.19)$$

We use data on futures prices as an approximation of the forward price. The approximation error is probably not important for monetary policy purposes given the usual relatively short times to expiration (see Appendix for a discussion).

Equations (2.18) and (2.19) can be used to fit the distribution by choosing the parameters to minimize, for instance, squared price errors. There are $3n - 1$ parameters: $\left\{\bar{b}^j + \sigma_{db}^j, \sigma_{bb}^j\right\}_{j=1}^{n}$ (since only the sum of $\bar{b}^j$ and σ_{db}^j enters) and $\{\alpha^j\}_{j=1}^{n-1}$ (since $\sum_{j=1}^{n} \alpha^j = 1$). We can use data on the forward price of the bond and at least $3n - 2$ options with different strike prices to solve for these parameters with some numerical method.

With only one normal distribution ($n = 1$), (2.19) can be used directly in (2.18) to get Black's [13] version of the Blacks-Scholes formula for options on bond futures

$$C(X) = e^{-i(t,\tau)(\tau-t)} \left[B(t, \tau, T)\, \Phi\left(\frac{\ln[B(t, \tau, T)/X] + \frac{\sigma_{bb}}{2}}{\sqrt{\sigma_{bb}}}\right) - X\Phi\left(\frac{\ln[B(t, \tau, T)/X] - \frac{\sigma_{bb}}{2}}{\sqrt{\sigma_{bb}}}\right)\right]. \quad (2.20)$$

This formula can be used to back out the implied variance (σ_{bb}). The implied variance is often very different at different strike prices (a "smile" is a common pattern), which indicates that the assumption of a single normal distribution is too crude. An alternative approach to fitting

[11] From (2.1) we have for $n = 1$ that $\bar{b} = -(T - \tau)\mathrm{E}_t i(\tau, T)$, $\sigma_{bb} = (T - \tau)^2 \mathrm{Var}_t i(\tau, T)$, $\bar{d} = \mathrm{E}_t d(t, \tau)$, $\sigma_{dd} = \mathrm{Var}_t d(t, \tau)$, and $\sigma_{db} = (T - \tau)\mathrm{Cov}_t [d(t, \tau), -i(\tau, T)]$.

distributions, used by Shimko [80] and Malz [58], is to fit a function $\sigma_{bb}(X)$ to the implied volatilities in Figure 2.3.b. This function is used in (2.20), and then (2.15) or (2.16) are applied.

It is straightforward to show (see Appendix) that we could equally well arrive at (2.18) by solving (2.14) with the risk-neutral pdf $\widetilde{h}(b)$ given by the true marginal distribution of b, $h(b) = \sum_{j=1}^{n} \alpha^j \phi\left(b; \bar{b}^j, \sigma_{bb}^j\right)$, except that the means are adjusted to equal $\bar{b}^j + \sigma_{db}^j$ for each j,

$$\widetilde{h}(b) = \sum_{j=1}^{n} \alpha^j \phi\left(b; \bar{b}^j + \sigma_{db}^j, \sigma_{bb}^j\right). \tag{2.21}$$

We noted before that we can only identify the sum of $\bar{b}^j + \sigma_{db}^j$, not its individual components. Equation (2.21) shows that this is equivalent to saying that we can only uncover the risk-neutral distribution of the bond price.

It is clear that the true expectation of the future bond price, $\mathrm{E}_t B(\tau, T)$, is of the same form as (2.19) but with σ_{db}^j set to zero. If $\sigma_{db}^j = \sigma_{db}$ for all j, then $B(t, \tau, T) = \exp(\sigma_{db}) \mathrm{E}_t B(\tau, T)$, which we can use in (2.19) to get the forward rate,

$$f(t, \tau, T) = \mathrm{E}_t i(\tau, T) - \frac{\eta + \sigma_{db}}{T - \tau}. \tag{2.22}$$

Here η is a Jensen inequality term $\eta = \ln \mathrm{E}_t B(\tau, T) - \mathrm{E}_t \ln B(\tau, T)$. The second term in (2.22) is, of course, the forward term premium, $\varphi^f(t, \tau, T)$, in (2.6). We could back out σ_{db} if we knew the forward term premium, since η can be computed from the parameters in the risk-neutral distribution (see Appendix for details). The true distribution of the future interest rate is then obtained by shifting each normal pdf to the left by σ_{db}.[12]

To illustrate these methods, the solid curve in *Figure 2.2.c* shows a fitted single normal distribution ($n = 1$) for the options in Figure 2.2.a. It looks reasonably similar to the dotted curve, but it cannot capture the tendency to skewness. Therefore, in *Figure 2.2.d* we show the results from mixing two normal distributions ($n = 2$). The mixture pdf shows a considerable negative skewness indicating a perceived risk of a large increase in future DM interest rates, but not of a large decrease. It is clear that already two normals with different means and variances provide considerable flexibility in the distribution.

Although it is of interest to assess the uncertainty of this *point estimate* of the distribution, the literature has not suggested specific ways to do this. The most straightforward way is perhaps to plot the price errors of the options as in *Figure 2.3.a*, or the implied volatility ($\sqrt{\sigma_{bb}}$

[12] Equation (2.21) also shows that (2.19) can be written as $B(t, \tau, T) = \widetilde{\mathrm{E}}_t B(t', \tau, T)$, that is, the forward price "martingales" with respect to the risk-neutral distribution. This was first shown by Harrsion and Kreps [49].

backed out from (2.20)) for data and fitted prices as in *2.3.b*. Yet another way is to claim that a mixture of n log-normal distributions is a correct model (and not just a convenient interpolation technique), and that actual prices differ from theoretical prices with a random error term. In this case, minimizing the sum of the squared pricing errors is a non-linear least-squares estimation of the parameters in the distribution function. The estimated parameters are approximately (and asymptotically) normally distributed. We accordingly use a heteroskedastic-consistent estimator of the covariance matrix to account for heteroskedastic price errors (see Davidsson and MacKinnon [28], chapter 16), and apply the delta method to get the approximate 95% point-by-point confidence interval for the pdf in *Figure 2.3.c*. Alternatively, a confidence interval can be computed by Monte Carlo simulations.

As an illustration of how this type of data can be used, we take a closer look at the Bunds market around Bundesbank's March 2, 1994, announcement of a very high M3 growth for January. *Figure 2.3.d* shows the fitted pdfs (mixture of two normals) of the June 1994 Bund options for the trade dates February 23 and March 4. The pdfs show that the market expected higher future interest rates after the announcement, but also that it became more uncertain about the future rates.[13]

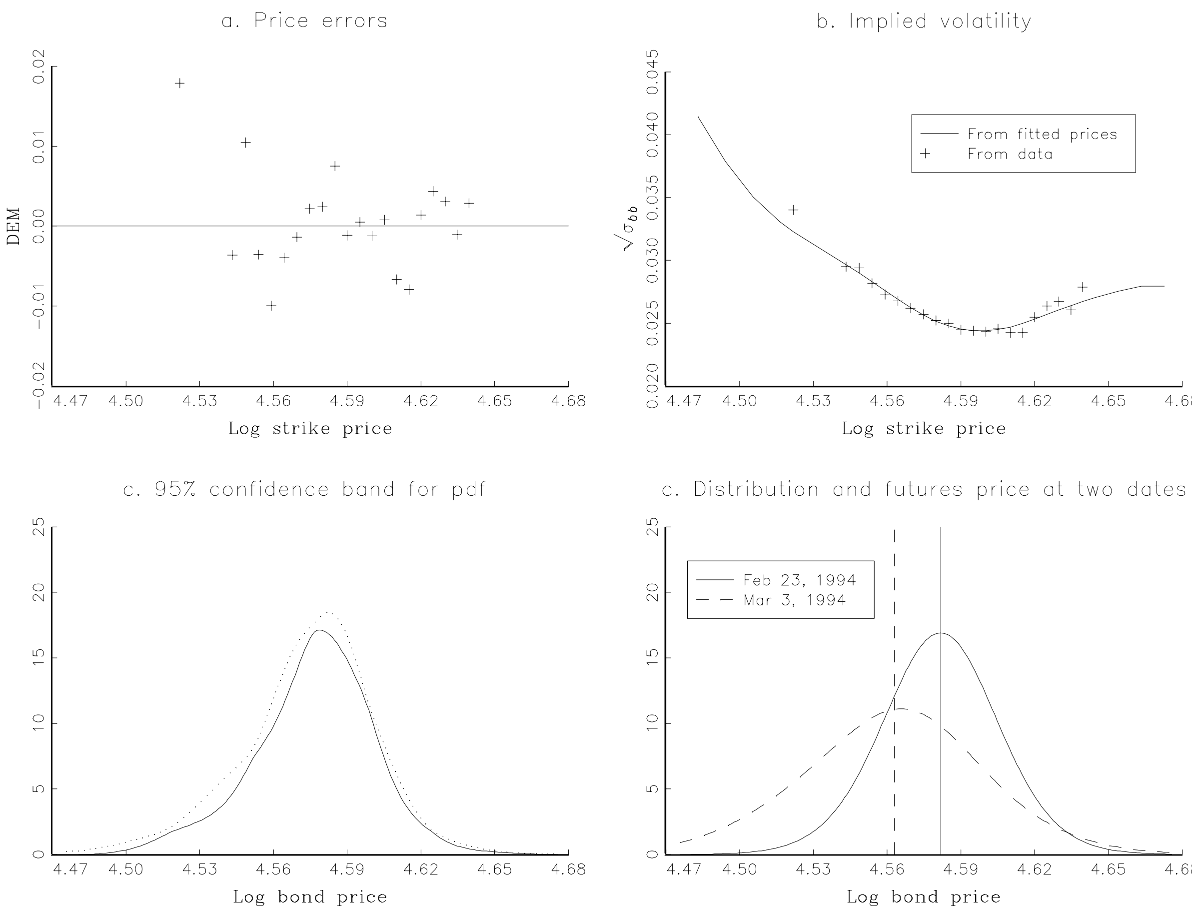
The above figures use data on exchange traded options. They are available from the exchanges (even historical series are often available). Data on over-the-counter (OTC) options may be better, though. OTC options are usually very liquid, have a fixed time to expiration (rather than a fixed date of expiration), and are often expressed in terms of the "delta" which means that the effective grid of strike prices changes with the futures price. Unfortunately data on OTC options are usually proprietary. (See Malz [57] for an insightful discussion.)

2.4 The expectations hypothesis of interest rates

Forward rates can (aside from the Jensen inequality term in (2.8)) be interpreted as risk-neutral means of future short interest rates. Above, we have discussed risk-neutral probability distributions. In order to translate those into corresponding "true" subjective means and distributions, we need to assess the size and variability of the relevant risk premia.

[13] For closely related approaches, see Neuhaus [70], Bahra [6], and Bank for International Settlements [10]. Mizrach [68] also fits the parameters in a mixture of lognormals, but by minimizing the percentage error in implied volatilities and in a Monte Carlo setting which allows pricing more complex options. Other approaches, which instead parameterize the stochastic process of the underlying asset, have also proved to be very useful. Bates [12] and Malz [57] estimate the parameters jump-diffusion models (including the implied "realignment" frequency and size) by minimizing squared option price errors. Campa and Chang [16] discusses ERM credibility by comparing currency option prices with the maximum value consistent with zero probability of moving outside the current band.

Figure 2.3: Price errors, implied volatility and confidence band for fitted distributions from Bund options at trade date April 6, 1994. Comparison of distributions at trade dates February 23 and March 3, 1994.



We relax the assumption of rational expectations in (2.6), and rewrite it as

$$f(t,\tau,T) = \mathrm{E}_t^m i(\tau,T) + \varphi^f(t,\tau,T), \tag{2.23}$$

where $\mathrm{E}_t^m i(\tau,T)$ denotes the interest rate expectations of the market (which may differ from the rational expectations $\mathrm{E}_t i(\tau,T)$). In order to calculate expected interest rates from the forward rate or the true distribution from the risk-neutral distribution (see (2.22)), we need to know the forward term premium, $\varphi^f(t,\tau,T)$. According to (2.8), the term premium is the conditional covariances of the future interest rate and the stochastic discount factor (plus a Jensen's inequality term).

We have no direct measurement of this (potentially time-varying) covariance, and even ex post data is of limited use since the stochastic discount factor is not observable. It has unfortunately proved to be very hard to explain (U.S. ex post) term premia by either utility based asset pricing models or various proxies for risk.[14] Other authors have instead estimated term premia directly by regressing ex post premia on past information. The fitted premia are typically rather persistent, which could possibly mean that changes in forward rates over a not too long period could signal changes in expected interest rates.[15]

Most of the evidence on risk premia, however, comes from regressions of the ex post future $\tau - t$-period changes in $T - \tau$-period interest rates on the forward spread,

$$i(\tau, T) - i(t, t+T-\tau) = \alpha + \beta \left[f(t, \tau, T) - i(t, t+T-\tau) \right] + \varepsilon(T). \tag{2.24}$$

The mean term premium should be captured by $\alpha + (1-\beta)\mathrm{E}[i(t, t+T-\tau) - f(t, \tau, T)]$, while the regression coefficient, β, can tell us something about to what extent the risk premium changes over time. The Appendix shows that the regression coefficient in a large sample is (see also Fama [35] and Froot [41])

$$\beta = 1 - \frac{\sigma(\sigma+\rho)}{1+\sigma^2+2\rho\sigma} + \gamma, \qquad \text{where} \tag{2.25}$$

$$\sigma = \frac{\mathrm{Std}\left(\varphi^f\right)}{\mathrm{Std}\left(\mathrm{E}_t^m \Delta i\right)}, \; \rho = \mathrm{Corr}\left(\mathrm{E}_t^m \Delta i, \varphi^f\right), \text{ and} \tag{2.26}$$

$$\gamma = \frac{\mathrm{Cov}\left[\left(\mathrm{E}_t - \mathrm{E}_t^m\right)\Delta i, \mathrm{E}_t^m \Delta i + \varphi^f\right]}{\mathrm{Var}\left(\mathrm{E}_t^m \Delta i + \varphi^f\right)}. \tag{2.27}$$

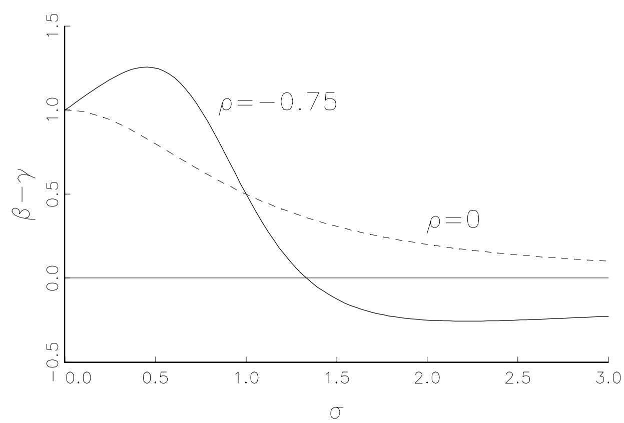
In (2.26) and (2.27), Δi is short for $i(\tau, T) - i(t, t+T-\tau)$ and φ^f for $\varphi^f(t, \tau, T)$. The second term in (2.25) captures the effect of the risk premium. The third term (γ) captures any systematic expectations errors.

Figure 2.4 shows how the expectations corrected regression coefficient ($\beta - \gamma$) depends on the relative volatility of the term premium and expected interest change (σ) and their correlation (ρ). A regression coefficient of unity could be due to either a constant term premium ($\sigma = 0$), or to a particular combination of relative volatility and correlation ($\rho = -\sigma$), which makes the forward spread an unbiased predictor. When the correlation is zero, the regression coefficient decreases monotonically with σ, since an increasing number of the movements in the forward rate are then due to the risk premium. A coefficient below a half is only possible when the term

[14] See, for instance, Mankiw [59], and Backus, Gregory and Zin [4]. For a discussion of the implications of the observed (ex post) term premium for the stochastic discount factor, see Backus, Foresi, and Zin [5].

[15] See, for instance, Dahlquist [26].

Figure 2.4: Expectations corrected regression coefficient ($\beta - \gamma$) as a function of the relative standard deviation (σ) and correlation (ρ).



premium is more volatile than the expected interest rate change ($\sigma > 1$), and a coefficient below zero also requires a negative correlation ($\rho < 0$).[16]

U.S. data often show β values between zero and one for very short maturities, around zero for maturities between 3 to 9 months, and often relatively close to one for longer maturities.[17] Fama and Bliss [37] and Campbell and Shiller [17] report that β tends to increase with the forecasting horizon (keeping the maturity constant), at least for horizons over a year.

The specification of the regression equation also matters, especially for long maturities: β is typically negative if the left hand side is the change in long rates, but much closer to one if it is an average of future short rates. The β estimates are typically much closer to one if the regression is expressed in levels rather than differences.[18] Even if this is disregarded, the point estimates for long maturities differ a lot between studies. The results in Jorion and Mishkin [52] suggest one possible explanation. They estimate ρ to be strongly negative. If this is true, then it is clear from Figure 2.4 that even small changes in σ around one can lead large changes in the estimated β.

Froot [41] uses a long sample of survey data on interest rate expectations. The results indicate that risk premia are important for the 3-month and 12-month maturities, but not for

[16] The curve for $\rho > 0$ is very similar to the curve for $\rho = 0$.
[17] See, for instance, Shiller [78] and Rudebusch [75] for overviews.
[18] See, for instance, Campbell and Shiller [17], Hardouvelis [47], Campbell and Shiller [17] and Bekaert, Hodrick, and Marshall [8]. The regression on levels is the flip side of the tracking of forward rate and future interest rate level documented by, for instance, Shiller [78].

really long maturities. On the other hand, there seems to be significant systematic expectations errors ($\gamma < 0$) for the long maturities which explain the negative β estimates in ex post data. We cannot, of course, tell whether these expectation errors are due to a small sample (for instance, a "peso problem") or to truly irrational expectations.[19]

We should expect β values to depend on the shocks to the economy and the type of monetary policy. It is sometimes argued that the U.S. Fed's policy makes interest rates changes almost unforecastable.[20] For a given volatility of the term premium, this should (in most cases) lead to a low value of β (by increasing σ). This argument gets some support from non-U.S. ex post data, which often show more predictable interest rate changes and a stronger support for the expectations hypothesis.[21]

It is difficult to draw clear conclusions from this empirical evidence. In some cases, the expectations hypothesis may be a reasonable approximation, with either a negligible or at least a constant risk premium. In other cases, in particular for large short-term interest rate movements, the size and variability of term premia may make these important. The current state of knowledge seems, unfortunately, to leave no other choice but to asses in each situation—using available information from different sources, experience and good judgement—whether a particular shift in the forward rate curve or the risk neutral distribution is due to a shift in expectations or risk premia.

3 Exchange rate expectations

3.1 Exchange rate expectations and risk premia

Let

$$\delta(t,T) = \frac{s(T) - s(t)}{T - t}$$

denote the domestic *currency depreciation rate* between dates t and $T > t$, where $s(t) = \ln S(t)$ and $S(t)$ is the exchange rate, the domestic currency price of one unit of foreign currency.

The expected future currency depreciation rate between dates τ and T is given by

$$\mathrm{E}_t \delta(\tau, T) = \mathrm{E}_t i(\tau, T) - \mathrm{E}_t i^*(\tau, T) - \varphi^s(t, \tau, T),$$

[19] Bekaert, Hodrick, and Marshall [8] argue that thirty-five years of monthly data can indeed be a small sample for a series with the kind of persistence and regime switches that appear to characterize U.S. interest rates. See Evans [31] for an overview of the small sample problems of the β estimator.

[20] For models of how monetary policy can affect (2.24), see, for instance, Rudebusch [75], Balduzzi, Bertola, and Foresi [7], and Kugler [53].

[21] See, for instance, Jorion and Mishkin [52], Hardouvelis [47], and Gerlach and Smets [44]. For some international survey evidence, see, for instance, MacDonald and Macmillan [56] who report mixed results for a short (3 years) U.K. sample.

where $\varphi^s(t,\tau,T)$ is the nominal forward foreign exchange risk premium and * denotes foreign currency variables. By (2.6) we can write this as

$$\mathrm{E}_t\delta(\tau,T) = [f(t,\tau,T) - f^*(t,\tau,T)] - \left[\varphi^f(t,\tau,T) - \varphi^{f*}(t,\tau,T)\right] - \varphi^s(t,\tau,T), \tag{3.1}$$

the difference between the domestic and foreign nominal forward rates, less the difference between the domestic and foreign nominal forward term premium, and less the forward foreign exchange risk premium. A special case of this is the expected future currency depreciation from the trade date t to $T > t$,

$$\mathrm{E}_t\delta(t,T) = [i(t,T) - i^*(t,T)] - \varphi^s(t,t,T), \tag{3.2}$$

the difference between the domestic and foreign nominal interest rates, less the spot foreign exchange risk premium.

Under the assumption of SDF pricing and lognormality, the forward foreign exchange risk premium can be written

$$\begin{aligned}\varphi^s(t,\tau,T) &= \frac{1}{2}(T-\tau)\left[\mathrm{Var}_t i(\tau,T) - \mathrm{Var}_t i^*(\tau,T)\right] \\ &\quad + \frac{1}{2}(T-\tau)\mathrm{Var}_t\delta(\tau,T) + \mathrm{Cov}_t\left[d(\tau,T),\delta(\tau,T)\right].\end{aligned} \tag{3.3}$$

The spot foreign exchange risk premium in (3.2) is given by

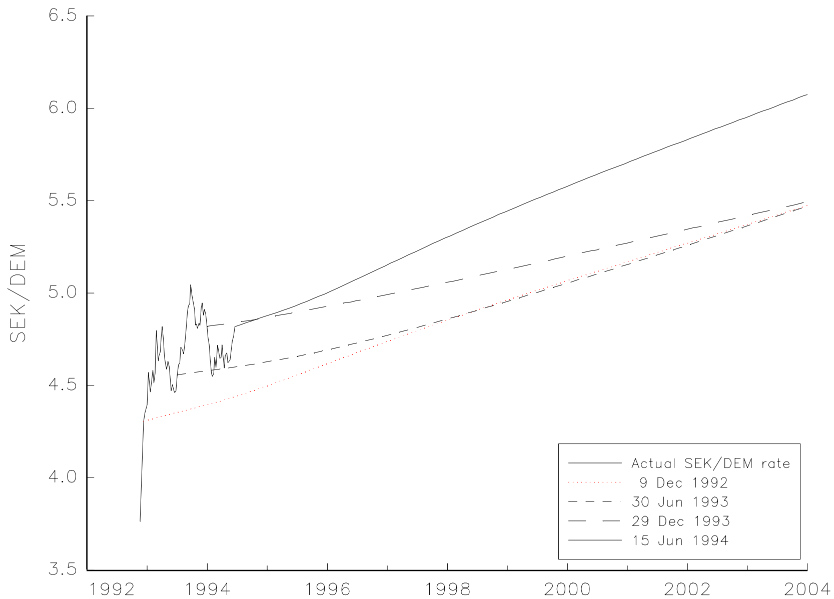
$$\varphi^s(t,t,T) = \frac{1}{2}(T-t)\mathrm{Var}_t\delta(t,T) + \mathrm{Cov}_t\left[d(t,T),\delta(t,T)\right]. \tag{3.4}$$

The first term on the right hand side in (3.4) is a Jensen inequality term. In the second term, note that the currency depreciation rate is the nominal excess rate of return on foreign currency bonds over domestic currency bonds. The term is therefore a standard covariance between the nominal discount factor and a nominal excess return (see Svensson [86] for details on the interpretation).

3.2 Instruments

Since cross-currency comparability is crucial in the above expressions, it is natural to use Euro currency rates and, for shorter maturities, Euro deposit rates, forward interest rates, and forward exchange rates. In particular, FRA's can be used instead of implicit forward interest rates. For longer maturities, interest swap rates are arguably a good data source. For shorter maturities,

Figure 3.1: Expected SEK/DEM exchange rate, December 9, 1992-June 15, 1994.



the forward exchange rates can be used instead of domestic and foreign interest rates in (3.2), since by covered interest parity

$$i(t,T) - i^{*}(t,T) = \frac{\ln F(t,T) - s(t)}{T-t}, \tag{3.5}$$

where $F(t,T)$ is the forward exchange rate, the price in domestic currency, determined at the trade date t, to be paid at date T for delivery of one unit of foreign currency at date T.

Figure 3.1 shows the actual Swedish krona/Deutsche mark exchange rate since November 18, 1992, the day before the krona was floated, and the expected future exchange rate (under the assumption of a negligible foreign exchange risk premium) as of four different trade dates from December 9, 1992, to June 15, 1994 (implicit spot interest rates from Swedish and German Treasury Bills and Government bonds have been used). The krona depreciated more rapidly than was expected on December 9, 1992. In spite of the large depreciation that had occurred by June 15, 1994, further depreciation was expected.

3.3 Extracting risk-neutral distributions from exchange rate options

Under SDF pricing the exchange rate satisfies

$$S\left(t\right) = \mathrm{E}_{t} D\left(t,\tau\right) S\left(\tau\right) e^{i^{*}(t,\tau)(\tau-t)}, \tag{3.6}$$

since investing $S(t)$ units of domestic currency in t at the foreign interest rate $i^*(t,\tau)$ gives a payoff of $S(\tau)\exp[i^*(t,\tau)(\tau-t)]$ units of domestic currency in τ.

As in Section 2.3, we assume that the log stochastic discount factor, $d(t,\tau)$, and the log asset price, $s(\tau)$, is distributed as a mixture of n normal distributions: just substitute s for b everywhere in (2.17).[22] We also keep the assumption that $\bar{d}^j = \bar{d}$ and $\sigma^j_{dd} = \sigma_{dd}$ for all j. Equations (3.6) and (3.5) then give that the forward exchange rate, $F(t,\tau)$, is as in (2.19) with s substituted for b. Similarly, the call option price is as in (2.18). The difference between the risk-neutral and the objective distribution is then related to the spot foreign exchange risk premium (3.4).

Many exchange rate options are American (also de facto). The possibility of early exercise of an American option makes the pricing problem harder: we have to specify not only the terminal distribution, but the distribution at all earlier dates, and to use numerical methods. However, Chaudhury and Wei [20] and Melick and Thomas [64] derive useful bounds for American call $(C^A(X))$ and put $(P^A(X))$ options on futures (assuming continuous trading).[23] They are

$$
\begin{aligned}
\underline{C}^A(X) &= \max\left[\widetilde{\mathrm{E}}_t S(\tau) - X, C(X)\right] && (3.7)\\
\bar{C}^A(X) &= e^{i(t,\tau)(\tau-t)} C(X) && (3.8)\\
\underline{P}^A(X) &= \max\left[X - \widetilde{\mathrm{E}}_t S(\tau), P(X)\right] && (3.9)\\
\bar{P}^A(X) &= e^{i(t,\tau)(\tau-t)} P(X), && (3.10)
\end{aligned}
$$

where $C(X)$ and $P(X)$ are the European call and put prices which satisfy

$$
P(X) = C(X) + e^{-i(t,\tau)(\tau-t)}[X - F(t,\tau)] \tag{3.11}
$$

(for a futures option, the discount factor multiplies both X and $F(t,\tau)$).

With a mixture of log-normals, $\widetilde{\mathrm{E}}_t S(\tau)$ is given by (2.19), and $C(X)$ by (2.18). The lower call bound converges towards the European call option from above as the strike prices increase. In contrast, the lower put bound starts at the European put option and converges towards $X - \widetilde{\mathrm{E}}_t S(\tau)$ as the strike prices increase. Increased volatility means that both lower bounds will stay close to the European option even for high strike prices, since the volatility increases the European call price. Both upper bounds are simply the European option times $e^{i(t,\tau)(\tau-t)}$.

[22] Substitute s, $\bar{s}^j$,σ^j_{ss}, and σ^j_{ds} for b, $\bar{b}^j$, σ^j_{bb}, and σ^j_{db}.

[23] Most options on bond futures expire at the same date as the futures contract expires. Then European options on bonds and on bond futures are equivalent. But American options on bonds and bond futures are then still different. Therefore the bounds reported are only valid for American options on bond futures.

These bounds are very narrow for typical times to expiration: the maximum width of both bands are $100\left[e^{i(t,\tau)(\tau-t)}-1\right] \approx 100\ i(t,\tau)(\tau-t)$ percent of the European option price. For short times to expiration, it may therefore be reasonable to assume that these options are European.

Melick and Thomas [64] use a more precise approach by treating the actual option prices as weighted averages of the lower and upper bounds, and by estimating two weights (ω^1,ω^2) along with the parameters in the risk-neutral distribution.[24] They suggest using ω^1 for in-the-money options and ω^2 for out-of-the-money options:

$$C^A(X) = w^c \underline{C}^A(X) + (1-w^c)\bar{C}^A(X), \tag{3.12}$$

$$P^A(X) = w^p \underline{P}^A(X) + (1-w^p)\bar{P}^A(X), \text{ where} \tag{3.13}$$

$$\begin{bmatrix} w^c \\ w^p \end{bmatrix} = \begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix} \text{ if } X < F \text{ and } \begin{bmatrix} w^c \\ w^p \end{bmatrix} = \begin{bmatrix} \omega^2 \\ \omega^1 \end{bmatrix} \text{ if } X > F. \tag{3.14}$$

A high weight on the lower bound means low chance of early exercise: the market expects uncertainty about the future exchange rate to be resolved far in the future.

Leahy and Thomas [54] apply this approach to options (expiring December 9, 1995) on Canadian dollar futures before and after the Quebec sovereignty referendum. *Figure 3.2.a* shows the available call and put options three days before the referendum. We follow Leahy and Thomas and fit the distribution of the future exchange rate as a mixture of three log-normal distributions. The result is shown by the solid curve in *Figure 3.2.c*.[25] The price errors for the options, illustrated by implied volatilities in *Figure 3.2.b*, are small but tend to be larger (in absolute size) for options that are very out-of-the-money. The figure actually has two curves, but they are very difficult to distinguish: one is based on estimates where we have used the correction for American-style option of Melick and Thomas, the other one estimates where we pretend that the options are European-style. The correction for American-style is not important, at least not compared to the pricing errors (which possibly reflect some other type of misspecification).[26]

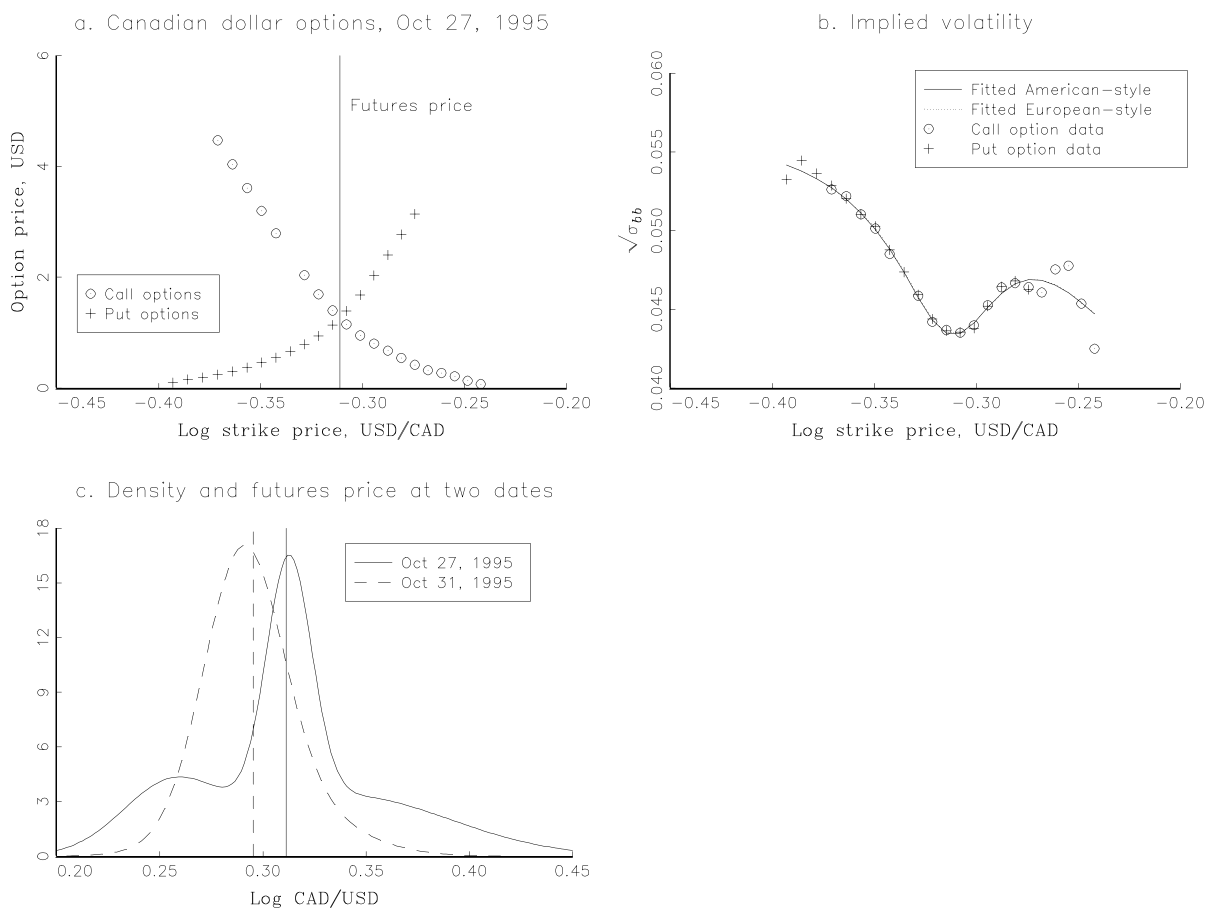
The dashed curve in *Figure 3.2.c* shows the fitted density function on the day after the referendum. A comparison of the two curves underscores the finding of Leahy and Thomas [54] that the "no" vote restored confidence in the Canadian dollar, both in terms of the overall

[24] Another common approach is to convert prices of American options into artificial prices of European options, as discussed in, for instance, Bates [11]. Mizrach [68] uses simulations of a parametric exchange rate process for pricing American options.

[25] There are some minor differences between our results and those in Leahy and Thomas [54]. One reason is that we include the price error of the futures price (an undiscounted option with zero strike price) in the loss function. Another reason is, of course, different optimization algorithms.

[26] With only about 1.5 months to expiration, with a 10 percent interest rate the maximum width of the bands between the bounds for an America option is only about 1.2 percent of the price of a European option.

Figure 3.2: Risk-neutral pdfs for December 1995 Canadian dollar futures as of October 27 and 31, 1995. Log Canadian dollar price of one U.S. dollar.



uncertainty and in terms of the expected future value.

When estimating implied distributions, we have noticed several practical problems, especially with a mixture of three normal distributions (rather than two). First, some estimates produce "spikes" in the density function, which often seem to be related to odd options prices. Second, the loss function seems to be very flat or even non-monotonic in some dimensions. In particular, it seems safer to apply an explicit grid search over the probabilities (α^j), rather than to exclusively rely on a gradient method for all parameters.[27]

If the states have reasonably clear interpretations, as in the Quebec case, it might be effi-

[27] Bahra [6] notes the problems with spikes, and discusses it at more length. Bates [12] finds signs of a flat loss function.

cient to estimate the distributions of future interest rates (log bond prices) and exchange rates simultaneously, by constraining them to have the same state probabilities (α^j).

3.4 Uncovered interest parity

Figures 3.1 shows (aside from a Jensen-inequality term) a riskneutral means and 3.2 shows risk-neutral distributions. As for interest rate expectations, in order to compute "true" subjective means and distributions, we need to assess the relevant risk premia. These premia appear in equation (3.1). Relaxing the assumption of rational expectations, we can write the equation as

$$f(t,\tau,T) - f^*(t,\tau,T) = \mathrm{E}_t^m \frac{s(T) - s(\tau)}{T-\tau} + \varphi^f(t,\tau,T) - \varphi^{f*}(t,\tau,T) + \varphi^s(t,\tau,T). \qquad (3.15)$$

Unfortunately, explaining exchange risk premia in terms of natural risk factors seems to be as hard as explaining term premia.[28] Several authors estimate exchange risk directly, by assuming rational expectations and regressing future exchange rates on current information. Similarly to the term premium, the fitted exchange risk premia are often persistent and significantly different from zero.[29]

Most of the knowledge about exchange rate risk premia comes from regressing the ex post exchange rate depreciation on the forward interest rate differential

$$s(T) - s(\tau) = \alpha + \beta(T-\tau)\left[f(t,\tau,T) - f^*(t,\tau,T)\right] + \varepsilon(T). \qquad (3.16)$$

This regression gives the same type of β coefficient as in (2.25)-(2.27) with $s(T) - s(\tau)$ substituted for Δi, and $(T-\tau)(\varphi^f - \varphi^{f*} + \varphi^s)$ for φ^f. Many of the problems with (2.24) carry over to (3.16), often with added strength since exchange rates tend to be very persistent and have big jumps.

Unfortunately, almost all studies have used $\tau = t$ (depreciation from now to T) rather than $t < \tau < T$, and concern the U.S. dollar. The typical result for a typical data set (major floating exchange rates like USD/DM, 1970s to 1990s) is significantly negative estimates of β.[30] This seems to hold for most forecasting horizons, but seems to be somewhat sensitive to the sample period (with the 1970s giving higher values than the 1980s). The UIP hypothesis seems to have better support in ERM cross rates than in many dollar rates, though.[31]

[28] See, for instance, Backus, Gregory, and Telmer [3] and Backus, Foresi, and Telmer [2]. Lewis [55] and Engel [30] provide good overviews.
[29] See, for instance, Cheung [23] and Canova and Marrinan [19].
[30] See Hodrick [51] and Engel [30] for overviews.
[31] See, for instance, Flood and Rose [38]. For a model of how monetary policy can affect (3.16), see McCallum [61].

Frankel and Froot [43] and [40] use survey data of expected changes in JPY, DM, GBP, and SFR against the U.S. dollar over the next one to twelve months. Correcting for systematic expectations errors in the sample, the β coefficients are in most cases between 0.8 and 1.2. Somewhat lower values (between zero and one) was obtained by Frankel and Chinn [39] from a survey of the expected changes in seventeen exchange rates. Unfortunately, these surveys cover only some three to eight years.

The empirical evidence in favor of uncovered interest rate parity is generally weak, especially for floating exchange rates and U.S. dollar rates, although survey data is more supportive of UIP. As for interest rate expectations, the current state of knowledge seems to leave no other option than to evaluate each particular shift of curves and distributions with the help of available information and use judgement to assess whether the shift is due to changes in expectations or risk premia.[32]

4 Inflation expectations

4.1 Inflation expectations and risk premia

Let $i(t,T)$ and $r(t,T)$ denote nominal and real interest rates, respectively, and let

$$\pi(t,T) = \frac{p(T) - p(t)}{T - t}$$

be the inflation rate between dates t and T. The expected inflation rate between future dates $\tau \geq t$ and $T > \tau$, $\mathrm{E}_t\pi(\tau,T)$, is given by

$$\mathrm{E}_t\pi(\tau,T) = \mathrm{E}_t i(\tau,T) - \mathrm{E}_t r(\tau,T) - \varphi^{\pi}(t,\tau,T),$$

where $\varphi^{\pi}(t,\tau,T)$ denotes the forward inflation risk premium. With (2.6) we can express the expected future inflation rate as

$$\mathrm{E}_t\pi(\tau,T) = f(t,\tau,T) - g(t,\tau,T) - \left[\varphi^{f}(t,\tau,T) - \varphi^{g}(t,\tau,T)\right] - \varphi^{\pi}(t,\tau,T), \qquad (4.1)$$

where $g(t,\tau,T)$ and $\varphi^{g}(t,\tau,T)$ denote the real forward interest rate and the real forward term premium, respectively. A special case of this is

$$\mathrm{E}_t\pi(t,T) = i(t,T) - r(t,T) - \varphi^{\pi}(t,t,T),$$

[32] One part of the literature, not covered here, has extracted exchange rate expectations for fixed exchange rate regimes in order to assess realignment expectations. Some approaches do not assume uncovered interest rate parity, for instance Svensson [83] and Campa and Chang [16]. Other do rely on uncovered interest rate parity, for instance the drift adjustment method in Rose and Svensson [73] and Svensson [85]. Peso problems are inherent in exchange rate target zones, making empirical tests of UIP less conclusive. Theoretical arguments in Svensson [84] indicate that UIP may be a reasonable approximation for narrow target zones.

where $\varphi^{\pi}(t,t,T)$ is the spot inflation risk premium.

Under the assumption of SDF pricing and lognormality, the forward inflation risk premium can be written in a number of different ways, one of which is

$$\begin{aligned}\varphi^{\pi}(t,\tau,T) &= \frac{1}{2}(T-\tau)\left[\mathrm{Var}_t i(\tau,T) - \mathrm{Var}_t r(\tau,T)\right] \\ &+ \frac{1}{2}(T-\tau)\mathrm{Var}_t \pi(\tau,T) + \mathrm{Cov}_t\left[d(\tau,T),\pi(\tau,T)\right]. \end{aligned} \tag{4.2}$$

The first and second terms are Jensen inequality terms; the third is a familiar covariance between the nominal discount factor and the inflation rate, equivalent to the covariance between the nominal discount factor and the excess return on real bonds relative to nominal bonds (see Svensson [86] for further discussion).

With information or assumptions about the risk premia, the expected future inflation rates can then be inferred from actual or implied nominal and real forward rates according to (4.1).

4.2 Instruments

Using real interest rates requires market quotes on index-linked bonds. The U.K. has the most significant market for index linked debt, with considerable volumes and a wide variety of maturities. Australia, Canada and Sweden also have such markets, although with less volume, liquidity and fewer outstanding maturities.

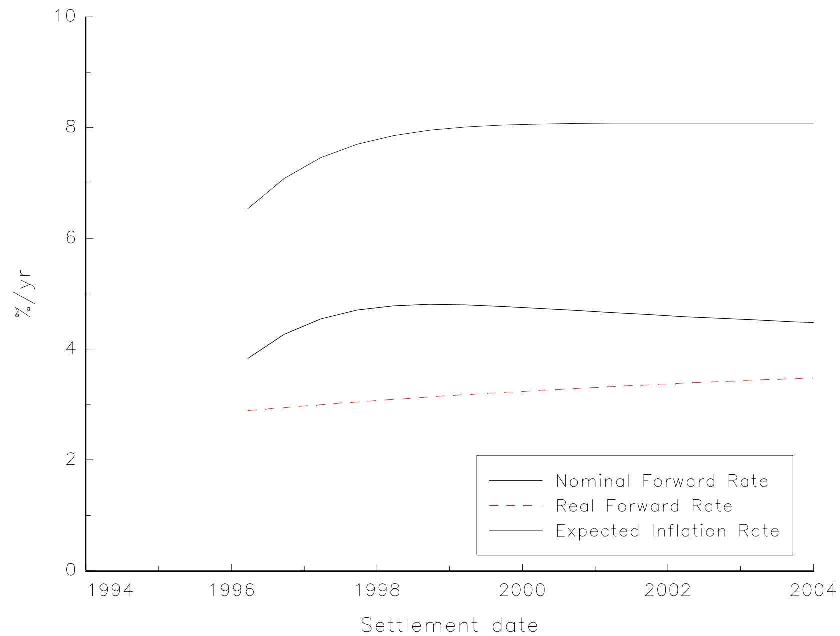
Unfortunately, details of the indexation and the tax treatment imply that indexed-linked bonds are rarely pure real bonds.[33]

Only the U.K. market has a liquidity and variety of maturities that invite more sophisticated estimates of implied real forward rates. The experience from the U.K. market indicates that simple functional forms for the forward rate are often sufficient (for instance an exponential, where all components except the first two in (2.10) are zero).

Since existing indexed-linked debt is mostly government debt, it seems suitable to use nominal implicit forward rates from government bills and bonds together with indexed-linked bond data. Figure 4.1 shows implied nominal and real forward rates, and expected future inflation rates (the latter under the assumption the combined risk premium in (4.1) is negligible) for the U.K. for the trade date March 22, 1995 (the estimates have been provided by Bank of England).

[33] The U.K. index-linked bonds are indexed to the price level 8 months prior to the coupon payment. Therefore they are in effect only partially indexed, which requires some adjustments and approximations in the estimation of the real yield curve. Evans [32] examines the theoretical and practical consequences of this. The former consist of working through the SDF pricing consequences of lagged indexation. A minimum lag in indexation, equal to the lag of about one month in the reporting of the CPI, is of course ideal.

Figure 4.1: Expected inflation rate, with nominal and real implied forward rates, United Kingdom, March 22, 1994



The curves are shown for settlements for two years and longer; for various reasons it is problematic to estimate real forward rates from U.K. index-linked bonds for settlements below two years.[34] The real forward rate curve is upward sloping and of a simple shape. The resulting expected inflation curve, "the term structure of inflation expectations," has a maximum of almost 5 percent per year in mid 1998.

Bank of England estimates show that the implied real forward rates (for settlements beyond two years) are much more stable than implied nominal forward rates, and that the real forward rate curve has a very simple shape. Often it is actually just a horizontal line. This lends some support for the frequent assumption that real forward rates (beyond some minimum time to settlement) are roughly constant, which assumption often has to be made for countries without indexed bonds or with a very thin market for these.

[34] Market quotes of longer real interest rates (and long implied real forward rates) are less variable than nominal interest rates (and implied forward rates). With sticky short-term inflation (expectations), short real interest rates should be as variable as short nominal interest rates, and hence be more variable than long real rates. It would be interesting to see whether this could be confirmed by relible quotes on short term real bonds.

4.3 Extracting risk-neutral distributions of inflation expectations

As far as we know there are no options traded on future inflation or real interest rates. Therefore, the method discussed cannot be applied directly. However, we can speculate on what could be done if at least real interest rate options were available. Therefore, suppose we have time t prices on options for both nominal and real bonds, with expiry and settlement date τ and maturity date T. These can indeed, under restrictive assumptions, be used to extract the implied risk-neutral distribution of future inflation *expectations*, $\mathrm{E}_t\pi\left(\tau, T\right)$. (In order to extract the distribution of future inflation, we would need options on inflation. These distributions should have the same mean, but the distribution of expectations should have a smaller variance.)

First, assume that the joint distribution of $i\left(\tau, T\right)$ and $r\left(\tau, T\right)$ is a mixture of bivariate normal distributions similar to (2.17), and that the inflation risk premium is small. In practice, we would estimate the marginal distributions of $i\left(\tau, T\right)$ and $r\left(\tau, T\right)$ jointly as discussed in Section 3.3. Second, assume some values for the conditional covariances between $i\left(\tau, T\right)$ and $r\left(\tau, T\right)$, denoted σ_{ir}^{j}, $j = 1, .., n$, for instance based on historical data. The implied risk-neutral distribution of $\mathrm{E}_\tau\pi\left(\tau, T\right) = i\left(\tau, T\right) - r\left(\tau, T\right)$ is then a mixture of n normal distributions with means $\bar{i}^j - \bar{r}^j$, variances $\sigma_{ii}^{j} + \sigma_{rr}^{j} - 2\sigma_{ir}^{j}$, and weights α^j.

4.4 The Fisher hypothesis

We relax the assumption of rational expectations in (4.1) and rewrite it as

$$f(t, \tau, T) = \mathrm{E}_t^m\pi(\tau, T) + g(t, \tau, T) + \varphi^f(t, \tau, T) - \varphi^g(t, \tau, T) + \varphi^\pi(t, \tau, T). \tag{4.3}$$

We need to control for both the real interest rate and the risk premia in order to calculate inflation expectations from nominal interest rate. Evidence from survey studies of inflation expectations suggests that the sum of the real interest rate and the risk premia has important medium-term movements.[35] While the real interest rate is at the core of most theoretical explorations of asset pricing models, relatively little attention has been paid to its time series properties and to the inflation risk premium.[36]

Similar to the expectations hypothesis and the uncovered interest parity, a common approach is to regress ex post inflation on nominal interest rates

$$\pi(t, T) - \pi(t-1, T-1) = \alpha + \beta\left[i(t, T) - i(t-1, T-1)\right] + \epsilon\left(T\right), \tag{4.4}$$

[35] See, for instance, Darin and Hetzel [27], Pennacchi [71], and Söderlind [81].
[36] See, for instance, Chan [21] for a study of the inflation risk premium.

which gives the same type of β coefficient as in (2.25)-(2.27) with change in inflation substituted for Δi, and the change in the sum of the real interest rate and risk premia for φ^f.

Mishkin [67] finds β between -0.3 and 0.3 for three months inflation rates on monthly U.S. ex post data for 1953 to 1990. He argues that the corresponding regression in levels gives regression coefficients close to one only for subsamples with clear trends in both inflation and interest rates. Söderlind [81] finds β values around 0.15 for one year inflation rates for both U.S. ex post and survey data (semi-annual data, 1953 to 1995). He argues that the level regressions differ between subsamples not so much because of different trends, but because of the different volatility of the real interest rate (where the early 1980s is the most pronounced outlier). The difference between rational expectations (based on ex post data) and market expectations (from surveys) is important for the level regression, but not for (4.4).

Mishkin [66] and Jorion and Mishkin [52] estimate a slightly modified (4.4) on monthly data for a number of OECD countries over 1973 to mid 1980. The coefficients are typically between 0 and 0.5 for short horizons, but increase with the horizon. Söderlind [81] estimates (4.4) on inflation expectations calculated from U.K. indexed-linked bonds for 1982-1995 and gets β close to one for forecasting horizons between one and four years (with similar results for levels).[37]

We believe there is little evidence of constant risk premia, at least for short forecasting horizons. However, the levels of nominal interest rates and inflation expectations seem to track each other fairly well over the medium-term horizon. This lends some support for using graphs like Figure 4.1 for extracting medium and long-term inflation expectations. Again, given the state of knowledge, judgement and available information must be used to assess whether a particular shift in the curves indicates shifting risk premia or shifting expectations.

5 Conclusions

There has been considerable progress in the last few years with regard to techniques to extract market expectations about interest rates, exchange rates and inflation from asset prices. These techniques have become feasible due to more developed and deeper financial markets, increased international financial integration, and new financial instruments. The methods discussed in this paper are being increasingly used for monetary policy purposes by an increasing number of central banks.

Suitable uses for these techniques include the following (see, for instance, Svensson [87] and

[37] For a model of how monetary policy can affect (4.4), see Söderlind [82].

[88] and Bahra [6] for more extensive discussion): Market expectations for short interest rates for the next 6-9 month, say, can be seen as market expectations of future monetary policy, when a short interest rate is used as the instrument. With such short horizons, the term premia are likely to be relatively small, so forward rate curves for settlements up to 6-9 months should be useful. Estimated probability distributions within the same horizon will also be useful in indicating the degree of market uncertainty about monetary policy in the near future.

In small open economies with floating exchange rates, the exchange rate channel is an important part of the monetary transmission mechanism. Indeed, some central banks use so-called monetary conditions indices (MCIs) that weigh interest rates and exchange rates together to measure the impact of monetary policy (see Gerlach and Smets [45] for discussion and references). Then it is natural to extract exchange rate expectations for the same horizon, and use those together with interest rate expectations to construct expected future MCIs. Again, estimated exchange rate distributions will indicate the uncertainty in the market's exchange rate expectations, which together with the distribution of interest rate expectations allow estimates of the distribution of future MCIs. For as short horizons as 6-9 months, the foreign exchange risk premium is arguably likely to be small, and the difference between the risk-neutral and 'true' subjective distributions will not be large.

In the medium and long term, say beyond 2-3 years, interest rate expectations, especially if there are sufficiently liquid indexed bonds, will be useful as indicating inflation expectations. These are always of interest to central banks, but particularly so if the bank has an explicit inflation target, since the difference between inflation expectations and the inflation target is an obvious measure of the (inverse) "credibility" of the monetary policy regime. Since there is some empirical support for the expectations hypothesis for such horizons, forward rates from nominal and indexed bonds should contain useful information. Still, sizeable inflation risk premia cannot be excluded, and any user and interpreter of such information must beware of possible shifts in both forward term premia and inflation risk premia. As always, it is advantageous to compare with other estimates of inflation expectations for the same horizon, for instance from surveys or from commercial forecasters.

For these horizons, estimated exchange rate expectations can also to some extent be interpreted as indicating the degree of credibility of the monetary regime, a large expected depreciation obviously likely to be associated with high inflation expectations and low "credibility". However, the dismal empirical support for uncovered interest rate parity warns against strong conclusions from moderate changes in extracted exchange rate expectations.

To our knowledge, options on interest rates or exchange rates are not traded for such long times to expiration, so extracted distributions are not available for these horizons.

For fixed exchange rate regimes, or exchange rate target zones, exchange rate expectations for all horizons are of particular relevance as indicators of the credibility of the regime.

We believe it is warranted to issue a warning about the *operational* use of measures of market expectations in monetary policy: Central banks should not react mechanically to such measures. For instance, the central bank's instrument should not be adjusted mechanically to "target" the difference between nominal interest rates or real interest rates (cf. Hetzel [50]), the difference between long nominal interest rates and some normal level of interest rates (cf. Goodfriend [46]), or the commercial forecasters' consensus forecast of inflation or nominal GDP (cf. Hall and Mankiw [48]). Nor should the central bank set short interest rates so as to fulfill the interest rate expectations that are manifested in the short end of a forward rate curve. Intuitively, if central banks target measures of market expectations with a mechanical feedback rule, and market expectations take this feedback into account, we have a situation which can be described as the central bank chasing the market and the market simultaneously chasing the central bank. The result may be that no equilibrium exists, or that there is a multiplicity of equilibria (see Woodford [92] and Bernanke and Woodford [9] for formal arguments).

In addition, a monetary policy that is based on the presumption that financial markets are always rational is bound to be problematic in situations when these markets are irrational (for shorter or longer periods), for instance, by overreacting. This is also an argument against mechanical central bank reactions to market expectations, and in favor of a monetary policy that is designed to work also in situations of market irrationality.

Thus, as argued in Svensson [90], a monetary policy of explicit inflation targeting, as in New Zealand, Canada, U.K., Sweden, Finland and Australia, does not imply that the central bank can regard *outside* inflation forecasts or expectations as an intermediate target that should be brought in line with the inflation target. Instead the central bank's internal "*structural*" (or "conditional") inflation forecast becomes an intermediate target, such that the central bank's instrument is chosen so that the forecast (conditional upon some structural model, the state of the economy, the instrument, and some judgement) is in line with the inflation target, or approaches the inflation target at an adequate rate. In such an internal forecast, market expectations may be one of many important information inputs, for instance for making assumptions about pricing behavior and wage settlements, without such expectations being mechanically targeted.

With regard to further work on the extraction of market expectations, the methods presented are fairly new, especially for extraction of distributions, and there is ample room for various refinements, operational innovations, and standardization of results. It is also clear that knowledge about the various risk premia is still very imperfect, especially for combinations of times to settlement and maturity that are relevant for monetary policy.

A wish list for financial instruments includes a full set of forward and options markets for nominal and indexed bonds, cross-currency exchange rates, and for the CPI index, for combinations of times to settlement and maturity that are relevant for monetary policy. Shiller [79] discusses new potential markets for allocating long-term risks, many of which would provide information of considerable interest to monetary policy.

A Interest rate expectations

A.1 Coupon bonds and yields

$B^c(t, T, c)$, the price at date t of a coupon bond with a constant coupon rate (constant flow) c and principal 1 that matures at date T

$$B^c(t, T, c) = c \int_{\tau=t}^{T} B(t, \tau) d\tau + B(t, T).$$

The corresponding bond yield (yield to maturity) $i(t, T, c)$ is the (unique) positive i that fulfills

$$\begin{aligned} B^c(t, T, c) &= c \int_{\tau=t}^{T} e^{-i(\tau - t)} d\tau + e^{-i(T-t)} \\ &= c \, \frac{1 - e^{-i(T-t)}}{i} + e^{-i(T-t)}. \end{aligned}$$

A coupon bond that pays a (discrete) coupon c at the dates τ_k, $k = 1, ..., K$, $t < t_1$, $t_k < t_{k+1}$, $\tau_K = T$

$$B^c(t, T, c) = c \sum_{k=1}^{K} B(t, \tau_k) + B(t, T).$$

The corresponding bond yield $i(t, T, c)$ is the (unique) positive i that fulfills

$$B^c(t, T, c) = c \sum_{k=1}^{K} e^{-i(\tau_k - t)} + e^{-i(T-t)}. \tag{A.1}$$

The terminology and conventions for different interest rates vary across markets and financial instruments, cf. Fabozzi [33] and Fage [34]. This paper uses the following terminology and definitions: Let i, i^e, and i^s denote, respectively, the continuously compounded annual interest rate i, effective annual interest rate i^e (effective annual yield), and the (simple) annual interest rate (bond-equivalent yield). They are related as

$$1 + i^e = e^i, \; i = \ln(1 + i^e), 1 + i^e = \left(1 + \frac{i^s}{n}\right)^n,$$

where (for a bond) $n \geq 1$ is the frequency of interest payments per year, (for UK and US bonds, $n = 2$), or (for a bill) $n = \frac{1}{m}$ where $m \leq 1$ is the maturity of the bill measured in years.

Cross-maturity comparisons are easiest for interest rates expressed as effective or continuously compounded annual rates. Definitions and algebra are normally easiest for continuously compounded rates.

A.2 SDF pricing under lognormality

Under (2.7), assume that $d(t,T) = D(t,T)$ is normal. Then

$$i(t,T) = -\frac{b(t,T)}{T-t} = -\frac{\mathrm{E}_t d(t,T) + \frac{1}{2}\mathrm{Var}_t d(t,T)}{T-t}$$

where we use the property that D lognormal implies $\mathrm{E}[D] = \exp\left[\mathrm{E}[\ln D] + \frac{1}{2}\mathrm{Var}[\ln D]\right]$. Then

$$\begin{aligned} f(t,\tau,T) &= -\frac{\left[\mathrm{E}_t d(t,T) + \frac{1}{2}\mathrm{Var}_t d(t,T)\right] - \left[\mathrm{E}_t d(t,\tau) + \frac{1}{2}\mathrm{Var}_t d(t,\tau)\right]}{T-\tau} \\ &= -\frac{\mathrm{E}_t d(\tau,T) + \frac{1}{2}\mathrm{Var}_t d(\tau,T) + \mathrm{Cov}_t\left[d(t,\tau), d(\tau,T)\right]}{T-\tau}, \end{aligned}$$

where we have used

$$\begin{aligned} d(t,T) &= d(t,\tau) + d(\tau,T) \\ \mathrm{Var}_t d(t,T) &= \mathrm{Var}_t d(t,\tau) + \mathrm{Var}_t d(\tau,T) + 2\mathrm{Cov}_t\left[d(t,\tau), d(\tau,T)\right]. \end{aligned}$$

We also have

$$\begin{aligned} i(\tau,T) &= -\frac{\mathrm{E}_\tau d(\tau,T) + \frac{1}{2}\mathrm{Var}_\tau d(\tau,T)}{T-\tau} \\ \mathrm{E}_t i(\tau,T) &= -\frac{\mathrm{E}_t\left[\mathrm{E}_\tau d(\tau,T) + \frac{1}{2}\mathrm{Var}_\tau d(\tau,T)\right]}{T-\tau} = -\frac{\mathrm{E}_t d(\tau,T) + \frac{1}{2}\mathrm{E}_t \mathrm{Var}_\tau d(\tau,T)}{T-\tau}. \end{aligned}$$

Since

$$\begin{aligned} \mathrm{Var}_t d(\tau,T) &= \mathrm{E}_t \mathrm{Var}_\tau d(\tau,T) + \mathrm{Var}_t \mathrm{E}_\tau d(\tau,T) \\ &= \mathrm{Var}_\tau d(\tau,T) + \mathrm{Var}_t \mathrm{E}_\tau d(\tau,T), \end{aligned}$$

(since $\mathrm{Var}_\tau d(\tau,T)$ is deterministic), and also

$$\begin{aligned} \mathrm{Var}_t \mathrm{E}_\tau d(\tau,T) &= \mathrm{Var}_t\left[\mathrm{E}_\tau d(\tau,T) + \frac{1}{2}\mathrm{Var}_\tau d(\tau,T)\right] \\ &= \mathrm{Var}_t\left[i(\tau,T)(T-\tau)\right], \end{aligned}$$

we have

$$\mathrm{Var}_\tau d(\tau,T) = \mathrm{Var}_t d(\tau,T) - \mathrm{Var}_t\left[i(\tau,T)(T-\tau)\right]. \tag{A.2}$$

We then get, in terms of Var_t,

$$\mathrm{E}_t i(\tau,T) = -\frac{\mathrm{E}_t d(\tau,T) + \frac{1}{2}\mathrm{Var}_t d(\tau,T)}{T-\tau} + \frac{1}{2}(T-\tau)\mathrm{Var}_t i(\tau,T). \tag{A.3}$$

With (A.3) we get

$$\begin{aligned}\varphi^f(t,\tau,T) &= f(t,\tau,T) - \mathrm{E}_t i(\tau,T) \\ &= -\frac{1}{2}(T-\tau)\mathrm{Var}_t i(\tau,T) - \frac{\mathrm{Cov}_t[d(t,\tau), d(\tau,T)]}{T-\tau} \\ &= -\frac{1}{2}(T-\tau)\mathrm{Var}_t i(\tau,T) - \mathrm{Cov}_t[d(t,\tau), f(t,\tau,T) - i(\tau,T)], \qquad \text{(A.4)}\end{aligned}$$

where we also have used

$$\begin{aligned}\frac{\mathrm{Cov}_t[d(t,\tau), d(\tau,T)]}{T-\tau} &= \frac{\mathrm{Cov}_t\left\{d(t,\tau), \mathrm{E}_\tau d(\tau,T) + [d(\tau,T) - \mathrm{E}_\tau d(\tau,T)]\right\}}{T-\tau} \\ &= \frac{\mathrm{Cov}_t\left[d(t,\tau), \mathrm{E}_\tau d(\tau,T)\right]}{T-\tau} = \mathrm{Cov}_t[d(t,\tau), -i(\tau,T)] \\ &= \mathrm{Cov}_t[d(t,\tau), f(t,\tau,T) - i(\tau,T)],\end{aligned}$$

since $\mathrm{E}_\tau d(\tau,T) = -i(\tau,T)(T-\tau) - \frac{1}{2}\mathrm{Var}_\tau d(\tau,T)$ and $\mathrm{Cov}_t[d(t,\tau), \mathrm{Var}_\tau d(\tau,T)] = 0$.

A.3 Estimating forward rates

Let $i(m)$ denote the spot rate $i(t, t+m)$ with time to maturity m, for a given trade date t. The extended Nelson and Siegel spot rate is given by

$$\begin{aligned}i(m;b) &= \beta_0 + \beta_1 \frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} + \beta_2\left(\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} - \exp\left(-\frac{m}{\tau_1}\right)\right) \\ &\quad + \beta_3\left(\frac{1-\exp\left(-\frac{m}{\tau_2}\right)}{\frac{m}{\tau_2}} - \exp\left(-\frac{m}{\tau_2}\right)\right).\end{aligned}$$

For a given trade date, let there be n coupon bonds (c_j, m_j, y_j, B_j^c), $j = 1, ..., n$, where c_j, m_j, y_j and B_j denote, respectively, the coupon rate, the time to maturity, the observed yield to maturity and the observed price of bond j, which is assumed to have a face value of 100 units of domestic currency. (The bond prices are computed from the yields to maturity according to (A.1), or vice versa.) For a given parameter vector b the estimated prices of the bonds, $B_j(b)$, are computed with the discount function $B(m;b)$ in (2.11) evaluating each coupon payment.

More precisely, for bonds with annual coupon payments, let τ_{jk}, $k = 1$, 2, ..., K_j, denote the times to the coupon payments on bond j, where K_j is the number of coupon payments. In the special case when m_j is an integer, we simply have $\tau_{jk} = k$ and $K_j = m_j$. In the general case we have

$$\begin{aligned}\tau_{jk} &= m_j - [m_j] + k - 1 \text{ and} \\ K_j &= [m_j] + 1,\end{aligned} \qquad \text{(A.5)}$$

where $[m_j]$ denotes the largest integer that is strictly smaller than m_j. The estimated price of each bond, $\hat{B}_j(b)$, is the present value of the bond when the coupon payments and the face value are priced with the discount function,

$$\hat{B}^c_j(b) \equiv \sum_{k=1}^{K_j} c_j B(\tau_{jk}; b) + B(\tau_{jK_j}; b),\ j = 1, ..., n. \tag{A.6}$$

For semiannual coupon payments, that is, for Britain and the United States, these relations are accordingly modified (see for instance Fage [34]).

When price errors are minimized, the observed price is assumed to differ from the estimated price by an error term, ϵ_j,

$$B^c_j = \hat{B}^c_j(b) + \epsilon_j. \tag{A.7}$$

The estimated prices are then fitted to the observed prices with Non-linear Least Squares, the General Method of Moments, or Maximum Likelihood. The estimates in this paper are Maximum Likelihood. The 95 percent confidence intervals have been computed with the delta method and are heteroskedasticity-consistent.[38]

When yield errors are minimized, the estimated yield to maturity for bond j, $\hat{Y}_j(b)$, is computed from the estimated bond price $\hat{B}_j(b)$ by solving (A.1). Although (A.1) is a non-linear higher-order equation with the same order as the number of coupon payments, it has only one real root and is easy to solve numerically, for instance with the standard Newton-Raphson algorithm (see Fage (1986)). The observed yield to maturity is assumed to differ from the estimated yield to maturity by an error term,

$$y_j = \hat{Y}_j(b) + \epsilon_j, \tag{A.8}$$

and the estimated yields to maturity are then fitted to the observed yields to maturity.

When the extended functional form (2.10) is used, it is obvious that perfect multicollinearity results if $\tau_1 = \tau_2$. Then only the sum $\beta_2 + \beta_3$ can be determined, not the individual components β_2 and β_3. In this case the two hump-shapes are located on top of each other, the model is over-parameterized, and the variance-covariance matrix cannot be computed. Thus, it is important that appropriate initial parameter values with $\tau_1 \neq \tau_2$ are given. If the estimation converges to the $\tau_1 = \tau_2$ the original Nelson and Siegel functional form is appropriate for that observation.

[38] Let $\hat{b}$ and $\hat{\Sigma}$ denote the estimates of the parameter vector b and its covariance matrix, respectively. The delta method implies that for the purpose of computing confidence intervals for the forward rate, the estimated forward rate $f(m; b)$ for a given settlement m is considered to be distributed as a Normal variable with mean $f(m; \hat{b})$ and covariance $\frac{\partial f(m;\hat{b})}{\partial b}' \hat{\Sigma} \frac{\partial f(m;\hat{b})}{\partial b}$, where $\frac{\partial f(m;\hat{b})}{\partial b}$ is the column vector of partial derivatives with respect to the parameters, etc.

One way to avoid that τ_1 and τ_2 become equal is to add a punishment term to the likelihood function.[39]

A.4 Extracting risk-neutral distributions from bond options

Conditional on state j the distribution of $d = \ln D(t,\tau)$ and $b = \ln B(\tau, T)$ is

$$\text{pdf}\left(\begin{bmatrix} d \\ b \end{bmatrix} |\text{state } j\right) = \phi\left(\begin{bmatrix} d \\ b \end{bmatrix}; \begin{bmatrix} \bar{d}^j \\ \bar{b}^j \end{bmatrix}, \begin{bmatrix} \sigma^j_{dd} & \sigma^j_{db} \\ \sigma^j_{db} & \sigma^j_{bb} \end{bmatrix}\right) = \phi\left(x; \mu^j, \Omega^j\right), \quad \text{(A.9)}$$

where we let $\phi(x; \mu^j, \Omega^j)$ denote a normal pdf of appropriate dimension (here two). The probability of state j is α^j. The joint pdf of j, d, and b is then $\alpha^j \phi\left(\begin{bmatrix} d & b \end{bmatrix}'; \mu^j, \Omega^j\right)$. Summing over the n states ("integrating out the state") gives the marginal pdf of d and b (2.17).

A bond forward contract written at time t stipulates that the investor gets one bond at τ (which will be worth $B(\tau, T)$), in return for the payment (at τ) of the forward price $B(t, \tau, T)$. This is like paying $\exp[-i(t,\tau)(\tau - t)] B(t, \tau, T)$ at t, or equivalently buying a bond maturing at T (worth $B(t, T)$), so the bond and forward price must satisfy

$$\begin{aligned}
B(t,T) &= e^{-i(t,\tau)(\tau-t)} B(t,\tau,T) \\
&= \mathrm{E}_t\left[D(t,\tau) B(\tau,T)\right] \\
&= \mathrm{E}_t e^{d+b} \\
&= \int_{b=-\infty}^{\infty} \int_{d=-\infty}^{\infty} e^{d+b} f(d,b)\, \mathrm{d}d\, \mathrm{d}b \\
&= \sum_{j=1}^{n} \alpha^j \left[\int_{b=-\infty}^{\infty} \int_{d=-\infty}^{\infty} e^{d+b} \phi\left(\begin{bmatrix} d & b \end{bmatrix}'; \mu^j, \Omega^j\right) \mathrm{d}d\, \mathrm{d}b\right] \\
&= \sum_{j=1}^{n} \alpha^j \left[\int_{z=-\infty}^{\infty} e^{z} \phi\left(z; \bar{d}^j + \bar{b}^j, \sigma^j_{dd} + \sigma^j_{bb} + 2\sigma^j_{db}\right) \mathrm{d}z\right] \\
&= \sum_{j=1}^{n} \alpha^j \exp\left(\bar{d}^j + \frac{\sigma^j_{dd}}{2}\right) \exp\left(\bar{b}^j + \frac{\sigma^j_{bb}}{2} + \sigma^j_{db}\right). \quad \text{(A.10)}
\end{aligned}$$

where $z = d + b$. If $T = \tau$, then $B(\tau, T)$ is unity, so the bond price must then satisfy

$$B(t,\tau) = e^{-i(t,\tau)(\tau-t)} = \sum_{j=1}^{n} \alpha^j \exp\left(\bar{d}^j + \frac{\sigma^j_{dd}}{2}\right). \quad \text{(A.11)}$$

If $\bar{d}^j = \bar{d}$ and $\sigma^j_{dd} = \sigma_{dd}$ for all j, then $B(t, \tau, T)$ in (A.10) can be rewritten by using (A.11) to give (2.19).

The same technique is used to price a European call option which allows the holder to buy a bond (with maturity date T) for the strike price X at the expiry date τ. To do that, note

[39] This was suggested by Jon Faust.

that the joint pdf $\alpha^j \phi\left(\left[\begin{array}{cc} d & b \end{array}\right]'; \mu^j, \Omega^j\right)$ can be written as $\alpha^j f(d|b,j) f(b|j)$ where $f(d|b,j)$ is the distribution of d conditional on b and j, and $f(b|j)$ the distribution of b conditional on j. Applying standard rules on the bivariate normal distribution, (A.9) these distributions being results in

$$\begin{aligned} f(d|b,j) &= \phi\left[d; \bar{d}^j + \frac{\sigma_{db}^j}{\sigma_{bb}^j}\left(b - \bar{b}^j\right), \sigma_{dd}^j - \frac{\left(\sigma_{db}^j\right)^2}{\sigma_{bb}^j}\right] \text{ and} \qquad (A.12) \\ f(b|j) &= \phi\left(b; \bar{b}^j, \sigma_{bb}^j\right). \qquad (A.13) \end{aligned}$$

Therefore, the call option price must satisfy

$$\begin{aligned} & \mathrm{E}_t \left\{D(t,\tau) \max\left[0, B(\tau,T) - X\right]\right\} \\ = & \int_{b=\ln X}^{\infty} \int_{d=-\infty}^{\infty} \left(e^b - X\right) e^d f(d,b) \,\mathrm{d}d \,\mathrm{d}b \\ = & \sum_{j=1}^{n} \alpha^j \left\{\int_{b=\ln X}^{\infty} \int_{d=-\infty}^{\infty} \left[e^b - X\right] e^d \phi\left(\left[\begin{array}{cc} d & b \end{array}\right]'; \mu^j, \Omega^j\right) \mathrm{d}d \,\mathrm{d}b\right\} \\ = & \sum_{j=1}^{n} \alpha^j \left\{\int_{b=\ln X}^{\infty} \left[e^b - X\right] \left[\int_{d=-\infty}^{\infty} e^d f(d|b,j) \,\mathrm{d}d\right] f(b|j) \,\mathrm{d}b\right\}. \qquad (A.14) \end{aligned}$$

It is straightforward to calculate the term within braces in (A.14) (see, for instance, Rubinstein [74]). One important step is to note that the innermost integral is the expected value of a variable, whose logarithm is normally distributed as in (A.12). We therefore have (by the rule $\mathrm{E}e^x = e^{\mathrm{E}x + \mathrm{Var}(x)/2}$ if x is normally distributed)

$$\begin{aligned} & \int_{d=-\infty}^{\infty} e^d f(d|b,j) \,\mathrm{d}d \\ = & \exp\left(\bar{d}^j + \frac{1}{2}\sigma_{dd}^j\right) \exp\left[\frac{\sigma_{db}^j}{\sigma_{bb}^j}\left(b - \bar{b}^j\right) - \frac{1}{2}\frac{\left(\sigma_{db}^j\right)^2}{\sigma_{bb}^j}\right]. \qquad (A.15) \end{aligned}$$

Consider the product of this term with $f(b|j)$. By (A.13) and (A.15) it is

$$\begin{aligned} & f(b|j) \int_{d=-\infty}^{\infty} e^d f(d|b,j) \,\mathrm{d}d \\ = & \exp\left(\bar{d}^j + \frac{1}{2}\sigma_{dd}^j\right) \exp\left[\frac{\sigma_{db}^j}{\sigma_{bb}^j}\left(b - \bar{b}^j\right) - \frac{1}{2}\frac{\left(\sigma_{db}^j\right)^2}{\sigma_{bb}^j}\right] \frac{1}{\sqrt{2\pi\sigma_{bb}^j}} \exp\left[-\frac{\left(b - \bar{b}^j\right)^2}{2\sigma_{bb}^j}\right] \\ = & \exp\left(\bar{d}^j + \frac{1}{2}\sigma_{dd}^j\right) \frac{1}{\sqrt{2\pi\sigma_{bb}^j}} \exp\left[-\frac{\left(b - \bar{b}^j - \sigma_{db}^j\right)^2}{2\sigma_{bb}^j}\right] \qquad (A.16) \end{aligned}$$

This means that the term within braces in (A.14) can be written as

$$\begin{aligned}
&\int_{b=\ln X}^{\infty}\left[e^{b}-X\right]\left[\int_{d=-\infty}^{\infty}e^{d}f\left(d|b,j\right)\mathrm{d}d\right]f\left(b|j\right)\mathrm{d}b \\
&= \exp\left(\bar{d}^{j}+\frac{1}{2}\sigma_{dd}^{j}\right)\int_{b=\ln X}^{\infty}\left[e^{b}-X\right]\phi\left(b;\bar{b}^{j}+\sigma_{db}^{j},\sigma_{bb}^{j}\right)\mathrm{d}b. \qquad \text{(A.17)}
\end{aligned}$$

The first term on the RHS is just discounting (and will equal $\exp\left[-i\left(t,\tau\right)\left(\tau-t\right)\right]$ if $\bar{d}^{j}=\bar{d}$ and $\sigma_{dd}^{j}=\sigma_{dd}$). The second term is a single integral, which equals the expected value of $\max\left(0,B-X\right)$ if the distribution of b is normal with mean $\bar{b}^{j}+\sigma_{db}^{j}$ and variance σ_{bb}^{j}. This can be thought of as a risk-neutral distribution, which substantiates the claim in (2.21).

Evaluating the integral in (A.17) is easy and the result is that the term within braces in (A.14) can be written as

$$\begin{aligned}
&\int_{b=\ln X}^{\infty}\left[e^{b}-X\right]\left[\int_{d=-\infty}^{\infty}e^{d}f\left(d|b,j\right)\mathrm{d}d\right]f\left(b|j\right)\mathrm{d}b \\
&= \exp\left(\bar{d}^{j}+\frac{\sigma_{dd}^{j}}{2}\right)\exp\left(\bar{b}^{j}+\frac{1}{2}\sigma_{bb}^{j}+\sigma_{db}^{j}\right)\Phi\left(\frac{\bar{b}^{j}+\sigma_{bb}^{j}+\sigma_{db}^{j}-\ln X}{\sqrt{\sigma_{bb}^{j}}}\right)- \\
&\quad \exp\left(\bar{d}^{j}+\frac{\sigma_{dd}^{j}}{2}\right)X\Phi\left(\frac{\bar{b}^{j}+\sigma_{db}^{j}-\ln X}{\sqrt{\sigma_{bb}^{j}}}\right). \qquad \text{(A.18)}
\end{aligned}$$

If $\bar{d}^{j}=\bar{d}$ and $\sigma_{dd}^{j}=\sigma_{dd}$, then by using (A.11) and (A.18) in (A.14) we get (2.18).

Expression (A.17) also shows that the risk-neutral expectation of $B\left(\tau,T\right)$,

$$\widetilde{\mathrm{E}}_{t}B\left(\tau,T\right)=\sum_{j=1}^{n}\alpha^{j}\left[\int_{b=-\infty}^{\infty}e^{b}\phi\left(\bar{b}^{j}+\sigma_{db}^{j},\sigma_{bb}^{j}\right)\mathrm{d}b\right]=\sum_{j=1}^{n}\alpha^{j}\exp\left(\bar{b}^{j}+\sigma_{db}^{j}+\frac{\sigma_{bb}^{j}}{2}\right), \qquad \text{(A.19)}$$

is identical with the forward price $B\left(t,\tau,T\right)$ in (2.19). This shows that the forward price can be thought of as a risk-neutral martingale process. We can also see from (A.19) that the true expectation, $\mathrm{E}_{t}B\left(\tau,T\right)$, is as in (A.19) but with σ_{db}^{j} set to zero.

Suppose that $\sigma_{db}^{j}=\sigma_{db}$ for all j. Then the forward interest rate is

$$f\left(t,\tau,T\right)=-\frac{\ln B\left(t,\tau,T\right)}{T-\tau}=-\frac{\sigma_{db}}{T-\tau}-\frac{1}{T-\tau}\ln\left\{\sum_{j=1}^{n}\alpha^{j}\exp\left(\bar{b}^{j}+\frac{\sigma_{bb}^{j}}{2}\right)\right\}. \qquad \text{(A.20)}$$

The term in curly braces equals $\mathrm{E}_{t}B\left(\tau,T\right)$, so

$$\begin{aligned}
f\left(t,\tau,T\right) &= -\frac{\ln \mathrm{E}_{t}B\left(\tau,T\right)}{T-\tau}-\frac{\sigma_{db}}{T-\tau} \\
&= \mathrm{E}_{t}i\left(\tau,T\right)-\frac{\eta+\sigma_{db}}{T-\tau}, \qquad \text{(A.21)}
\end{aligned}$$

where Jensen inequality term, η, fulfills

$$\begin{aligned} \eta &= \ln \mathrm{E}_t B(\tau, T) - \mathrm{E}_t \ln B(\tau, T) \\ &= \ln \left[\sum_{j=1}^{n} \alpha^j \exp\left(\bar{b}^j + \frac{\sigma_{bb}^j}{2} \right) \right] - \sum_{j=1}^{n} \alpha^j \bar{b}^j. \end{aligned} \tag{A.22}$$

We can solve for σ_{db} by using α^j, σ_{bb}^j and $\bar{b}^j + \sigma_{db}$ from the risk-neutral distribution and a value of the forward term premium, $\varphi^f = -(\eta + \sigma_{db}) / (T - \tau)$. Let $z^j = \bar{b}^j + \sigma_{db}$ by the fitted means of the risk-neutral distribution. Then

$$\begin{aligned} (T - \tau) \varphi^f &= -\ln \left[\sum_{j=1}^{n} \alpha^j \exp\left(\bar{b}^j + \frac{\sigma_{bb}^j}{2} \right) \right] + \sum_{j=1}^{n} \alpha^j \bar{b}^j - \sigma_{db} \\ &= -\ln \left[\sum_{j=1}^{n} \alpha^j \exp\left(z^j + \frac{\sigma_{bb}^j}{2} \right) \right] + \sum_{j=1}^{n} \alpha^j z^j - \sigma_{db}, \end{aligned} \tag{A.23}$$

which contains only one unknown, σ_{db}.

In the case of only one state ($n = 1$), (2.19) becomes

$$B(t, \tau, T) = \exp\left(\bar{b} + \sigma_{db} + \frac{\sigma_{bb}}{2} \right), \tag{A.24}$$

and (2.18) becomes

$$\begin{aligned} C(X) &= e^{-i(\tau - t)} \left[\exp\left(\bar{b} + \frac{1}{2}\sigma_{bb} + \sigma_{db} \right) \phi\left(\frac{\bar{b} + \sigma_{bb} + \sigma_{db} - \ln X}{\sqrt{\sigma_{bb}}} \right) \right] \\ &\quad - X \Phi\left(\frac{\bar{b} + \sigma_{db} - \ln X}{\sqrt{\sigma_{bb}}} \right). \end{aligned} \tag{A.25}$$

Using (A.24) in (A.25) gives (2.20). Moreover, ln $\mathrm{E}_t B(\tau, T) = \bar{b} + \frac{\sigma_{bb}}{2}$, so the Jensen term in (A.22) is $\frac{1}{2}\sigma_{bb}$.

A.5 Forward and futures prices

The forward contract is an agreement made at t to deliver an asset at τ (worth $S(\tau)$) against a payment made at τ of $F(t, \tau)$. The *forward price* must therefore satisfy

$$\begin{aligned} 0 &= \mathrm{E}_t \{ D(t, \tau) [S(\tau) - F(t, \tau)] \} \qquad \text{(A.26)} \\ F(t, \tau) &= \frac{\mathrm{E}_t D(t, \tau) S(\tau)}{\mathrm{E}_t D(t, \tau)} \\ &= e^{i(t,\tau)(\tau - t)} \mathrm{E}_t D(t, \tau) S(\tau). \end{aligned}$$

This will in general differ from the *futures price* $H(t, \tau)$ which is subject by "marking to market". This can be thought of as dividends equal to $H(t + j\Delta, \tau) - H(t + (j-1)\Delta, \tau)$,

$j = 1,..,k$, $\Delta = \frac{\tau - t}{k}$, paid to the owner of the contract at each date $t + j\Delta$. An arbitrage argument (see Cox, Ingersoll, and Ross [25]) can be used to show that the futures price must satisfy

$$H(t,\tau) = \mathrm{E}_t \left[\exp\left(\sum_{j=0}^{k-1} i(t + j\Delta, t + (j+1)\Delta)\Delta \right) D(t,\tau) S(\tau) \right]. \tag{A.27}$$

Equality of $F(t,\tau)$ and $H(t,\tau)$ would, for instance, follow from a deterministic interest rate where $i(t,\tau)(\tau - t) = \sum_{j=0}^{k-1} i(t + j\Delta, t + (j+1) + \Delta)\Delta$. The difference between the forward and future price is usually small for $\tau - t$ of a few months.

A.6 The expectations hypothesis of interest rates

Let $n = \tau - t$, $m = T - \tau$, and let $u(t+n)$ be the innovation in the interest rate and $\eta(t)$ be the expectation error of market participants

$$\begin{aligned} i(t+n, t+n+m) &= \mathrm{E}_t i(t+n, t+n+m) + u(t+n) \\ \mathrm{E}_t i(t+n, t+n+m) &= \mathrm{E}_t^m i(t+n, t+n+m) + \eta(t). \end{aligned}$$

The regression coefficient in (2.25) is then

$$\begin{aligned} \beta &= \frac{\mathrm{Cov}\left[i(t+n) - i(t), f(t) - i(t)\right]}{\mathrm{Var}\left[f(t) - i(t)\right]} \\ &= \frac{\mathrm{Cov}\left[\mathrm{E}_t \Delta_n i(t+n) + u(t+n), \mathrm{E}_t^m \Delta_n i(t+n) + \varphi^f(t)\right]}{\mathrm{Var}\left[\mathrm{E}_t^m \Delta_n i(t+n) + \varphi^f(t)\right]} \\ &= \frac{\mathrm{Cov}\left[\mathrm{E}_t^m \Delta_n i(t+n) + \eta(t) + u(t+n), \mathrm{E}_t^m \Delta_n i(t+n) + \varphi^f(t)\right]}{\mathrm{Var}\left[\mathrm{E}_t^m \Delta_n i(t+n) + \varphi^f(t)\right]} \\ &= \frac{\mathrm{Var}\left[\mathrm{E}_t^m \Delta_n i(t+n)\right] + \mathrm{Cov}\left[\mathrm{E}_t^m \Delta_n i(t+n), \varphi^f(t)\right] + \mathrm{Cov}\left[\eta(t), \mathrm{E}_t^m \Delta_n i(t+n) + \varphi^f(t)\right]}{\mathrm{Var}\left[\mathrm{E}_t^m \Delta_n i(t+n)\right] + \mathrm{Var}\left[\varphi^f(t)\right] + 2\mathrm{Cov}\left[\mathrm{E}_t^m \Delta_n i(t+n), \varphi^f(t)\right]}, \end{aligned}$$

since $u(t+n)$ is orthogonal to all information in t. Divide both numerator and denominator with $\mathrm{Var}(\mathrm{E}_t^m \Delta_n i(t+n))$ and use the definitions of σ, ρ, and γ in (2.26) and (2.27) to get (2.25).

B Exchange rate expectations

B.1 SDF pricing under lognormality

Assume SDF pricing and lognormality,

$$B^*(t,T) = \mathrm{E}_t D^*(t,T) = \mathrm{E}_t \left[D(t,T) \frac{S(T)}{S(t)} \right], \tag{B.1}$$

where $d^*(t,T) = d(t,T) + (T-t)\delta(t,T)$. In analogy with (A.3),

$$\mathrm{E}_t i^*(\tau,T) = -\frac{\mathrm{E}_t d^*(\tau,T) + \frac{1}{2}\mathrm{Var}_t d^*(\tau,T)}{T-\tau} + \frac{1}{2}(T-\tau)\mathrm{Var}_t i^*(\tau,T). \tag{B.2}$$

With (A.3) and (B.2) we get

$$\begin{aligned}
\varphi^s(t,\tau,T) &= \mathrm{E}_t i(\tau,T) - \mathrm{E}_t i^*(\tau,T) - \mathrm{E}_t \delta(\tau,T) \\
&= \mathrm{E}_t i(\tau,T) + \frac{\mathrm{E}_t d^*(\tau,T) + \frac{1}{2}\mathrm{Var}_t d^*(\tau,T)}{T-\tau} - \frac{1}{2}(T-\tau)\mathrm{Var}_t i^*(\tau,T) - \mathrm{E}_t \delta(\tau,T) \\
&= \mathrm{E}_t i(\tau,T) + \frac{\mathrm{E}_t\left[d(\tau,T) + (T-\tau)\delta(\tau,T)\right] + \frac{1}{2}\mathrm{Var}_t\left[d(\tau,T) + (T-\tau)\delta(\tau,T)\right]}{T-\tau} \\
&\quad - \frac{1}{2}(T-\tau)\mathrm{Var}_t i^*(\tau,T) - \mathrm{E}_t \delta(\tau,T) \\
&= \left(\mathrm{E}_t i(\tau,T) + \frac{\mathrm{E}_t d(\tau,T) + \frac{1}{2}\mathrm{Var}_t d(\tau,T)}{T-\tau}\right) \\
&\quad + \frac{1}{2}(T-\tau)\mathrm{Var}_t \delta(\tau,T) + \mathrm{Cov}_t\left[d(\tau,T), \delta(\tau,T)\right] - \frac{1}{2}(T-\tau)\mathrm{Var}_t i^*(\tau,T) \\
&= \frac{1}{2}(T-\tau)\left[\mathrm{Var}_t i(\tau,T) - \mathrm{Var}_t i^*(\tau,T)\right] \\
&\quad + \frac{1}{2}(T-\tau)\mathrm{Var}_t \delta(\tau,T) + \mathrm{Cov}_t\left[d(\tau,T), \delta(\tau,T)\right]. \qquad \text{(B.3)}
\end{aligned}$$

C Inflation expectations

Assume SDF pricing and lognormality. Let $d(t,T)$ be the log nominal discount factor. Then the log real discount factor, $q(t,T)$, fulfills

$$q(t,T) = d(t,T) + p(T) - p(t) = d(t,T) + (T-t)\pi(t,T).$$

Use, in analogy with (A.3),

$$\begin{aligned}
\mathrm{E}_t r(\tau,T) &= -\frac{\mathrm{E}_t q(\tau,T) + \frac{1}{2}\mathrm{Var}_t q(\tau,T)}{T-\tau} + \frac{1}{2}(T-\tau)\mathrm{Var}_t r(\tau,T) \qquad \text{(C.1)} \\
&= -\frac{\mathrm{E}_t\left[d(\tau,T) + (T-\tau)\pi(\tau,T)\right] + \frac{1}{2}\mathrm{Var}_t\left[d(\tau,T) + (T-\tau)\pi(\tau,T)\right]}{T-\tau} + \frac{1}{2}(T-\tau)\mathrm{Var}_t r(\tau,T).
\end{aligned}$$

With (A.3) and (C.1) we get

$$\begin{aligned}
\varphi^\pi(t,\tau,T) &= \mathrm{E}_t i(\tau,T) - \mathrm{E}_t r(\tau,T) - \mathrm{E}_t \pi(\tau,T) \\
&= \mathrm{E}_t i(\tau,T) + \frac{\mathrm{E}_t\left[d(\tau,T) + (T-\tau)\pi(\tau,T)\right] + \frac{1}{2}\mathrm{Var}_t\left[d(\tau,T) + (T-\tau)\pi(\tau,T)\right]}{T-\tau} \\
&\quad - \frac{1}{2}(T-\tau)\mathrm{Var}_t r(\tau,T) - \mathrm{E}_t \pi(\tau,T) \\
&= \left(\mathrm{E}_t i(\tau,T) + \frac{\mathrm{E}_t d(\tau,T) + \frac{1}{2}\mathrm{Var}_t d(\tau,T)}{T-\tau}\right)
\end{aligned}$$

$$
\begin{aligned}
& + \frac{1}{2}(T-\tau)\mathrm{Var}_t \pi(\tau,T) + \mathrm{Cov}_t\left[d(\tau,T),\pi(\tau,T)\right] - \frac{1}{2}(T-\tau)\mathrm{Var}_t r(\tau,T) \\
= & \frac{1}{2}(T-\tau)\left[\mathrm{Var}_t i(\tau,T) - \mathrm{Var}_t r(\tau,T)\right] \\
& + \frac{1}{2}(T-\tau)\mathrm{Var}_t \pi(\tau,T) + \mathrm{Cov}_t\left[d(\tau,T),\pi(\tau,T)\right] \qquad \text{(C.2)} \\
= & \frac{1}{2}(T-\tau)\left[\mathrm{Var}_t i(\tau,T) - \mathrm{Var}_t r(\tau,T)\right] \\
& - \frac{1}{2}(T-\tau)\mathrm{Var}_t \pi(\tau,T) + \mathrm{Cov}_t\left[q(\tau,T),\pi(\tau,T)\right]. \qquad \text{(C.3)}
\end{aligned}
$$

Note that expressing (C.2) with the real discount factor, $q(\tau,T)$, changes the sign on the $\mathrm{Var}_t\pi(t,T)$ term.

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