Appendix to

Monetary Policy with Judgment: Forecast Targeting

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Contents

A. Optimal policy under commitment with the deviation being an arbitrary stochastic process

Let the model equations for $t \geq 0$ be (2.1). A common special case is when the matrix $C = I$, but in general C need not be invertible. This system can be written

$$\tilde{C} \begin{bmatrix} X_{t+1} \\ \mathrm{E}_t x_{t+1} \\ \mathrm{E}_t i_{t+1} \end{bmatrix} = \tilde{A} \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} + \begin{bmatrix} z_{t+1} \\ 0 \end{bmatrix}, \tag{A.1}$$

where $\mathrm{E}_t q_{t+\tau} \equiv \int q_{t+\tau} d\Phi_t(\zeta^t)$ for any variable $q_{t+\tau}$ ($\tau \geq 0$), the matrices $\tilde{A}$ and $\tilde{C}$ are of dimension $(n_X + n_x) \times (n_X + n_x + n_i)$ and given by

$$\tilde{A} \equiv \begin{bmatrix} A & B \end{bmatrix} \equiv \begin{bmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \end{bmatrix}, \quad \tilde{C} \equiv \begin{bmatrix} I & 0 & 0 \\ 0 & C & 0 \end{bmatrix}.$$

where A and B are partitioned according to (2.3).

The target variables are defined by (2.5). The intertemporal loss function in period 0 is

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \delta^t L_t,$$

where the period loss function, (2.7), can be written as

$$L_t = \frac{1}{2} \begin{bmatrix} X_t' & x_t' & i_t' \end{bmatrix} D'WD \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}.$$

Consider minimizing this intertemporal loss function under once-and-for-all commitment in period $t = 0$, for given $X_0 = \bar{X}_0$. For convergence, when the variance of z_{t+1} is nonzero, I need $0 < \delta < 1$.

Variants of this problem are solved in Backus and Driffill [2], Currie and Levine [5], and Söderlind [20], when the deviation is an iid shock. The focus here is on the case when the deviation is an arbitrary stochastic process.

Construct the Lagrangian,

$$\begin{aligned} \mathcal{L}_0 &= \mathrm{E}_0 \sum_{t=0}^{\infty} \delta^t \left\{ L_t + \begin{bmatrix} \xi_{t+1}' & \Xi_t' \end{bmatrix} \left(\tilde{C} \begin{bmatrix} X_{t+1} \\ \mathrm{E}_t x_{t+1} \\ \mathrm{E}_t i_{t+1} \end{bmatrix} - \tilde{A} \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} - \begin{bmatrix} z_{t+1} \\ 0 \end{bmatrix} \right) \right\} \\ &\quad + \xi_0'(X_0 - \bar{X}_0)/\delta \\ &= \mathrm{E}_0 \sum_{t=0}^{\infty} \delta^t \left\{ L_t + \begin{bmatrix} \xi_{t+1}' & \Xi_t' \end{bmatrix} \left(\tilde{C} \begin{bmatrix} X_{t+1} \\ x_{t+1} \\ i_{t+1} \end{bmatrix} - \tilde{A} \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} - \begin{bmatrix} z_{t+1} \\ 0 \end{bmatrix} \right) \right\} \\ &\quad + \xi_0'(X_0 - \bar{X}_0)/\delta, \end{aligned}$$

where ξ_{t+1} and Ξ_t are vectors of n_X and n_x Lagrange multipliers of the upper and lower block, respectively, of (A.1). The law of iterated expectations has been used in the second equality, $\mathrm{E}_0\mathrm{E}_t = \mathrm{E}_0$ for $t \geq 0$. Note that Ξ_t is dated to emphasize that it depends on information available in period t.

For $t \geq 1$, the first-order conditions with respect to X_t, x_t and i_t can be written

$$\begin{bmatrix} X_t' & x_t' & i_t' \end{bmatrix} D'WD + \begin{bmatrix} \xi_t' & \Xi_{t-1}' \end{bmatrix} \frac{1}{\delta}\tilde{C} - \begin{bmatrix} \mathrm{E}_t\xi_{t+1}' & \Xi_t' \end{bmatrix} \tilde{A} = 0. \tag{A.2}$$

For $t = 0$, the first-order condition with respect to X_0, x_0, and i_0 can be written

$$\begin{bmatrix} X_t' & x_t' & i_t' \end{bmatrix} D'WD + \begin{bmatrix} \xi_t' & 0 \end{bmatrix} \frac{1}{\delta}\tilde{C} - \begin{bmatrix} \mathrm{E}_t\xi_{t+1}' & \Xi_t' \end{bmatrix} \tilde{A} = 0, \tag{A.3}$$

where $X_0 = \bar{X}_0$. In comparison with (A.2), a vector of zeros enters in place of Ξ_{-1}, since there is no constraint corresponding to the lower block of (A.1) for $t = -1$. By including a fictitious vector of Lagrange multipliers, Ξ_{-1}, equal to zero,

$$\Xi_{-1} = 0, \tag{A.4}$$

in (A.3), I can write the first-order conditions more compactly as (A.2) for $t \geq 0$ and (A.4).

The system of difference equations (A.2) has $n_X + n_x + n_i$ equations. The first n_X equations can be associated with the Lagrange multipliers ξ_t. Indeed, $-\xi_t/\delta$ can be interpreted as the total marginal losses in period t of the predetermined variables X_t (for $t = 0$, with given X_0, the equations determine ξ_0). They are forward-looking variables: Lagrange multipliers for predetermined variables are always forward-looking, whereas the Lagrange multipliers for the (equations for the) forward-looking variables are predetermined. The middle n_x equations are associated with the Lagrange multipliers Ξ_t. Indeed, $\Xi_t A_{22}$ can be interpreted as the total marginal losses in period t of the forward-looking variables, x_t. Also, $\Xi_t C$ can be seen as the marginal loss in period t of expectations $\mathrm{E}_t x_{t+1}$ of the forward-looking variables. The last n_i equations are the first-order equations for the vector of instruments. In the special case when the lower right $n_i \times n_i$ submatrix of $D'WD$ is of full rank, the instruments can be solved in terms of the other variables and eliminated from (A.2), leaving the first $n_X + n_x$ equations involving the Lagrange multipliers and the predetermined and forward-looking variables only.

Rewrite the $n_X + n_x + n_i$ first-order conditions as

$$\tilde{A}' \begin{bmatrix} \mathrm{E}_t \xi_{t+1} \\ \Xi_t \end{bmatrix} = D'WD \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} + \frac{1}{\delta} \tilde{C}' \begin{bmatrix} \xi_t \\ \Xi_{t-1} \end{bmatrix}. \tag{A.5}$$

They can be combined with the model equations (A.1) to get a system of $2(n_X + n_x) + n_i$ difference equations for $t \geq 0$,

$$\begin{bmatrix} \tilde{C} & 0 \\ 0 & \tilde{A}' \end{bmatrix} \begin{bmatrix} X_{t+1} \\ \mathrm{E}_t x_{t+1} \\ \mathrm{E}_t i_{t+1} \\ \hline \mathrm{E}_t \xi_{t+1} \\ \Xi_t \end{bmatrix} = \begin{bmatrix} \tilde{A} & 0 \\ D'WD & \frac{1}{\delta} \tilde{C}' \end{bmatrix} \begin{bmatrix} X_t \\ x_t \\ i_t \\ \hline \xi_t \\ \Xi_{t-1} \end{bmatrix} + \begin{bmatrix} z_{t+1} \\ 0 \\ \hline 0 \\ 0 \end{bmatrix}. \tag{A.6}$$

Here, X_t and Ξ_t are predetermined variables, and x_t, i_t, and ξ_t are non-predetermined variables.

This can be rearranged as the system

$$\mathcal{C} \begin{bmatrix} y_{1,t+1} \\ \mathrm{E}_t y_{2,t+1} \end{bmatrix} = \mathcal{M} \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} z_{t+1} \\ 0 \end{bmatrix} \\ 0 \end{bmatrix},$$

where

$$\mathcal{C} \equiv \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & A_{21}' & 0 & 0 & A_{11}' \\ 0 & A_{22}' & 0 & 0 & A_{12}' \\ 0 & B_2' & 0 & 0 & B_1' \end{bmatrix}, \tag{A.7}$$

$$y_{1t} \equiv \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix}, \qquad y_{2t} \equiv \begin{bmatrix} x_t \\ i_t \\ \xi_t \end{bmatrix}.$$

Thus, y_{1t} is a vector of $m_1 \equiv n_X + n_x$ predetermined variables, and y_{2t} is a vector of $m_2 \equiv n_x + n_i + n_X$ non-predetermined variables.

Under suitable assumptions (see appendix B), such a system has a unique solution, which can be written

$$
\begin{aligned}
y_{2t} &= F_1 y_{1t} + Z_t && \text{(A.8)} \\
y_{1,t+1} &= M_1 y_{1t} + N \mathrm{E}_t Z_{t+1} + P \mathrm{E}_t z_{t+1} + \begin{bmatrix} z_{t+1} - \mathrm{E}_t z_{t+1} \\ 0 \end{bmatrix}, && \text{(A.9)}
\end{aligned}
$$

where Z_t is an m_2-dimensional stochastic process given by

$$
Z_t \equiv \sum_{\tau=0}^{\infty} R_\tau \mathrm{E}_t z_{t+1+\tau} \equiv R z^t, \tag{A.10}
$$

where I can interpret R as a linear operator on $z^t \equiv \mathrm{E}_t(z'_{t+1}, z'_{t+2}, ...)'$.

In terms of the original variables, the solution for $t \geq 0$, given X_0 and $\Xi_{-1} = 0$, can be written

$$
\begin{aligned}
\begin{bmatrix} x_t \\ i_t \\ \xi_t \end{bmatrix} &= F_1 \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix} + R z^t \\
&\equiv F \begin{bmatrix} X_t \\ z^t \\ \Xi_{t-1} \end{bmatrix}, && \text{(A.11)} \\
\begin{bmatrix} X_{t+1} \\ \Xi_t \end{bmatrix} &= M_1 \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix} + N R \mathrm{E}_t z^{t+1} + P \mathrm{E}_t z_{t+1} + \begin{bmatrix} z_{t+1} - \mathrm{E}_t z_{t+1} \\ 0 \end{bmatrix} \\
&\equiv M \begin{bmatrix} X_t \\ z^t \\ \Xi_{t-1} \end{bmatrix} + \begin{bmatrix} z_{t+1} - \mathrm{E}_t z_{t+1} \\ 0 \end{bmatrix}, && \text{(A.12)}
\end{aligned}
$$

where F and M are linear operators. The details of the solution are derived in appendix B. The matrices F_1, M_1, N, P, and $\{R_\tau\}_{\tau=0}^{\infty}$—and thereby the linear operators M and F—depend on A, B, C, D, W, and δ, but are independent of the second and higher moments of the exogenous stochastic process $\{z_t\}_{t=1}^{\infty}$. This demonstrates the certainty equivalence of the commitment solution.[37]

If the commitment is once and for all and starts in period 0, $\Xi_{-1} = 0$. Commitment in a timeless perspective can be seen as corresponding to a situation where the lower block of (A.12) is restricted to apply also for previous periods. Then, Ξ_{t-1} is determined by

$$
\begin{aligned}
\Xi_{t-1} &= M_{121} X_{t-1} + M_{122} \Xi_{t-2} + N_2 \mathrm{E}_{t-1} Z_t + P_2 \mathrm{E}_{t-1} z_t \\
&= \sum_{\tau=0}^{\infty} {M_{122}}^{\tau} (M_{121} X_{t-1-\tau} + N_2 \mathrm{E}_{t-1-\tau} Z_{t-\tau} + P_2 \mathrm{E}_{t-1-\tau} z_{t-\tau}),
\end{aligned}
$$

[37] The middle block of (A.11) is the optimal *explicit instrument rule* for this problem, the instrument written as a function of predetermined and exogenous variables. Eliminating the Lagrange multipliers from (A.2) results in the optimal *targeting rule* for this problem, a consolidated optimal first-order condition for the target variables. See Svensson [25] on instrument and targeting rules, as well as the lecture notes Svensson [A7].

where M_1, N, and P are partitioned conformably with X_t and Ξ_{t-1}.

Alternatively, the commitment in a timeless perspective can be generated as optimization under commitment or discretion with a term added to the intertemporal loss function in period 0,

$$\mathrm{E}_0 \sum_{t=0}^{\infty} \delta^t L_t + \Xi_{-1} \frac{1}{\delta} C x_0,$$

where Ξ_{-1} is the Lagrange multiplier for the block of forward-looking equations from the optimization in period -1 (see Svensson and Woodford [30] and Svensson [25]).

In the standard case, when z_t is a vector of iid zero-mean shocks, I have $\mathrm{E}_t z_{t+1} \equiv 0$, $Z_t \equiv \mathrm{E}_t Z_{t+1} \equiv 0$, and $z^t \equiv 0$. Thus, the terms involving Z_t in (A.11) and (A.12) vanish.[38] Consequently, the effect of z_t being an arbitrary exogenous stochastic process shows up only in the addition of the terms involving Z_t and the corresponding matrices N, P, and $\{R_\tau\}_{\tau=0}^{\infty}$. Then, I can set $M \equiv M_1$ and $F \equiv F_1$, and

$$y_{1,t+1} = M y_{1t} + z_{t+1}.$$

Let Σ denote the variance-covariance matrix of the iid shocks z_{t+1}. Define the matrices $\bar{D}$ and $\bar{W}$ according to

$$Y_t = D \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix} = D \begin{bmatrix} I & 0 \\ F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} y_{1t} \equiv \bar{D} y_{1t},$$

$$L_t = \frac{1}{2} Y_t' W Y_t = \frac{1}{2} y_{1t}' \bar{D}' W \bar{D} y_{1t} \equiv \frac{1}{2} y_{1t}' \bar{W} y_{1t},$$

where $\bar{W}$ is symmetric and positive semidefinite. Then twice the minimum loss in period t will satisfy

$$\begin{aligned} y_{1t}' V y_{1t} + w &= \mathrm{E}_t \sum_{\tau=0}^{\infty} \delta^\tau y_{1,t+\tau}' \bar{W} y_{1,t+\tau} \\ &= y_{1t}' \bar{W} y_{1t} + \mathrm{E}_t \sum_{\tau=1}^{\infty} \delta^\tau y_{1,t+\tau}' \bar{W} y_{1,t+\tau} \\ &= y_{1t}' \bar{W} y_{1t} + \delta \mathrm{E}_t \mathrm{E}_{t+1} \sum_{\tau=0}^{\infty} \delta^\tau y_{1,t+1+\tau}' \bar{W} y_{1,t+1+\tau} \\ &= y_{1t}' \bar{W} y_{1t} + \delta \mathrm{E}_t (y_{1,t+1}' V y_{1,t+1} + w) \\ &= y_{1t}' \bar{W} y_{1t} + \delta (y_{1t}' M' V M y_{1t} + \mathrm{E}_t z_{1,t+1}' V z_{1,t+1} + w) \\ &= y_{1t}' \bar{W} y_{1t} + \delta y_{1t}' M' V M y_{1t} + \delta \mathrm{trace}(V\Sigma) + \delta w. \end{aligned}$$

[38] In the case when $\{z_t\}$ is an autoregressive process and can be written $z_{t+1} = \Psi z_t + \varepsilon_{t+1}$, where Ψ is a matrix and ε_t an iid zero-mean process, z_t can simply be included among the predetermined variable.

It follows that

$$w = \frac{\delta}{1-\delta}\mathrm{trace}(V\Sigma),$$

and that the matrix V satisfies the Lyapunov equation

$$V = \bar{W} + \delta M'VM. \tag{A.13}$$

It follows that when $\mathrm{trace}(V\Sigma)$ is nonzero, I must have $\delta < 1$ for the existence of an finite w.

I can use the relations $\mathrm{vec}(A+B) = \mathrm{vec}(A) + \mathrm{vec}(B)$ and $\mathrm{vec}(ABC) = (C' \otimes A)\ \mathrm{vec}(B)$ on (A.13) (where $\mathrm{vec}(A)$ denotes the vector of stacked column vectors of the matrix A, and $\otimes$ denotes the Kronecker product) which results in

$$\begin{aligned} \mathrm{vec}\,(V) &= \mathrm{vec}(\bar{W}) + \delta \mathrm{vec}\left(M'VM\right) \\ &= \mathrm{vec}(\bar{W}) + \delta\left(M' \otimes M'\right)\mathrm{vec}\,(V)\,. \end{aligned}$$

Solving for $\mathrm{vec}\,(V)$ gives

$$\mathrm{vec}\,(V) = \left[I - \delta\left(M' \otimes M'\right)\right]^{-1}\mathrm{vec}(\bar{W}). \tag{A.14}$$

A.1. No forward-looking variables

If there are no forward-looking variables, so $n_x = 0$, I have

$$\tilde{C}\begin{bmatrix} X_{t+1} \\ \mathrm{E}_t i_{t+1} \end{bmatrix} = \tilde{A}\begin{bmatrix} X_t \\ i_t \end{bmatrix} + z_{t+1}, \tag{A.15}$$

where the matrices $\tilde{A}$ and $\tilde{C}$ are of dimension $n_X \times (n_X + n_i)$ and given by

$$\tilde{A} \equiv \begin{bmatrix} A & B \end{bmatrix},\ \ \tilde{C} \equiv \begin{bmatrix} I & 0 \end{bmatrix}.$$

The period loss function is

$$L_t = \frac{1}{2}Y_t'WY_t \equiv \frac{1}{2}\begin{bmatrix} X_t' & i_t' \end{bmatrix} D'WD \begin{bmatrix} X_t \\ i_t \end{bmatrix}.$$

The $n_X + n_i$ first-order conditions can be written

$$\tilde{A}'\mathrm{E}_t\xi_{t+1} = D'WD\begin{bmatrix} X_t \\ i_t \end{bmatrix} + \frac{1}{\delta}\tilde{C}'\xi_t. \tag{A.16}$$

Combined with the model equations, I get a system of $2n_X + n_i$ difference equations for $t \geq 0$,

$$\begin{bmatrix} \tilde{C} & 0 \\ 0 & \tilde{A}' \end{bmatrix}\begin{bmatrix} X_{t+1} \\ \mathrm{E}_t i_{t+1} \\ \hline \mathrm{E}_t\xi_{t+1} \end{bmatrix} = \begin{bmatrix} \tilde{A} & 0 \\ D'WD & \frac{1}{\delta}\tilde{C}' \end{bmatrix}\begin{bmatrix} X_t \\ i_t \\ \hline \xi_t \end{bmatrix} + \begin{bmatrix} z_{t+1} \\ \hline 0 \end{bmatrix}.$$

Here, X_t are predetermined variables, and i_t and ξ_t are non-predetermined variables.

Under suitable assumptions, this system will have a unique solution for $t \geq 0$, given X_0, which can be written

$$\begin{aligned} \begin{bmatrix} i_t \\ \xi_t \end{bmatrix} &= F_1 X_t + R z^t, \\ X_{t+1} &= M_1 X_t + N_0 R z^t + z_{t+1}. \end{aligned}$$

When there are no forward-looking variables, X_{t+1} is directly determined by X_t, i_t, and z_{t+1} according to (2.1), so M_1 and N_0 are determined by A, B, and F_1 as

$$\begin{aligned} M_1 &\equiv A + BF_i, \\ N_0 &\equiv [B \;\; 0], \end{aligned}$$

where

$$F_1 \equiv \begin{bmatrix} F_i \\ F_\xi \end{bmatrix}$$

is partitioned conformably with i_t and ξ_t. In comparison with the general solution of (A.9), for the backward-looking case,

$$N_0 R z^t \equiv NR\mathrm{E}_t z^{t+1} + (P - I)\mathrm{E}_t z_{t+1}.$$

B. The solution of a system of difference equations with the deviation

In order to understand the term in the solution (A.10) and (A.11) that corresponds to the deviation, consider the system

$$\mathcal{C} \begin{bmatrix} y_{1,t+1} \\ \mathrm{E}_t y_{2,t+1} \end{bmatrix} = \mathcal{M} \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} + \begin{bmatrix} \theta_{t+1} \\ 0 \end{bmatrix} \qquad \text{(B.1)}$$

for $t \geq 0$; where y_{1t} is a vector of m_1 predetermined variables ($y_{1t} \equiv (X_t', \Xi_{t-1}')'$ and $m_1 = n_X + n_x$ in the previous section); y_{2t} is a vector of m_2 non-predetermined variables ($y_{2t} \equiv (x_t', i_t', \xi_t')'$ and $m_2 = n_x + n_i + n_X$ in the previous section); θ_t is an m_1-vector of stochastic processes ($\theta_t \equiv (z_t', 0')'$ in the previous section); and y_{10} is given.

By defining the m_2-vector of endogenous expectation errors, η_t, as

$$\eta_t \equiv y_{2t} - \mathrm{E}_t y_{2t},$$

(B.1) can be written in the form used in Sims [A6],

$$\Gamma_0 y_t = \Gamma_1 y_{t-1} + \Psi \theta_t + \Pi \eta_t,$$

where $y_t \equiv (y_{1t}', y_{2t}')'$. Sims shows that, under suitable assumptions, this system has a unique solution of the form

$$y_t = \Theta_1 y_{t-1} + \Theta_0 \theta_t + \Theta_y \sum_{\tau=0}^{\infty} \Theta_f^\tau \Theta_\theta \mathrm{E}_t \theta_{t+1+\tau},$$

where Θ_0 and Θ_1 are real matrices, Θ_y, Θ_f, and Θ_θ are complex matrices, and $\Theta_y \Theta_f^\tau \Theta_\theta$ for any integer $\tau \geq 0$ is a real matrix. These matrices can be calculated by his Matlab program Gensys, available at www.princeton.edu/∼sims. An advantage with Sims's approach is that one need not keep track of what variables are predetermined or nonpredetermined. An arguable disadvantage is that the determination of the expectational errors is somewhat complex.

Here, I prefer to keep close track of what variables are predetermined and nonpredetermined and therefore choose to derive the solution to (B.1) following a route closer to Klein [A4] than Sims [A6], but going beyond Klein in, as Sims, explicitly treating the case of θ_t being an arbitrary stochastic process rather than an autoregressive process. The solution will then be of the form

$$\begin{aligned}
y_{2t} &= F_1 y_{1t} + Z_t, \\
y_{1,t+1} &= M_1 y_{1t} + N \mathrm{E}_t Z_{t+1} + P \mathrm{E}_t \theta_{t+1} + (\theta_{t+1} - \mathrm{E}_t \theta_{t+1}), \\
Z_t &\equiv \sum_{\tau=0}^{\infty} R_\tau \mathrm{E}_t \theta_{t+1+\tau},
\end{aligned}$$

where F_1, M_1, N, P, and R_τ are real matrices to be determined.

Take the expectation conditional on information in period t and write the system as

$$\mathcal{C} \begin{bmatrix} \mathrm{E}_t y_{1,t+1} \\ \mathrm{E}_t y_{2,t+1} \end{bmatrix} = \mathcal{M} \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} + \begin{bmatrix} \mathrm{E}_t \theta_{t+1} \\ 0 \end{bmatrix}. \tag{B.2}$$

Following Klein [A4], Sims [A6], and Söderlind [20], I use the generalized Schur decomposition. This decomposition results in the square complex matrices Q, S, T, and Z such that

$$\mathcal{C} = Q' S Z', \tag{B.3}$$
$$\mathcal{M} = Q' T Z', \tag{B.4}$$

where Z' for a complex matrix denotes the complex conjugate transpose of Z (the transpose of the complex conjugate of Z).[39] The matrices Q and Z are unitary ($Q'Q = Z'Z = I$), and S and T are upper triangular (see Golub and van Loan [A2]). The decomposition is furthermore ordered so the block consisting of the stable generalized eigenvalues (the jth diagonal element of T divided

[39] Let the elements of the complex matrix Z be denoted $z_{jk} \equiv \mathrm{Re}(z_{jk}) + i\,\mathrm{Im}(z_{jk})$. Then the complex conjugate of the matrix Z is the matrix of elements $\bar{z}_{jk} \equiv \mathrm{Re}(z_{jk}) - i\,\mathrm{Im}(z_{jk})$.

by the jth diagonal element of S, $\lambda_j \equiv t_{jj}/s_{jj}$) comes first and the block of unstable generalized eigenvalues comes last.[40]

More precisely, I assume the *saddle-point property* emphasized by Blanchard and Kahn [A1]: The number of eigenvalues with modulus larger than unity equals the number of nonpredetermined variables. Thus, I assume that $|\lambda_j| > 1$ for $m_1+1 \leq j \leq m_1+m_2$ and $|\lambda_j| < 1$ for $1 \leq j \leq m_1$ (for an exogenous predetermined variable with a unit root, I can actually allow $|\lambda_j| = 1$ for some $1 \leq j \leq m_1$).

Define

$$\begin{bmatrix} \tilde{y}_{1t} \\ \tilde{y}_{2t} \end{bmatrix} \equiv Z' \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix}. \tag{B.5}$$

I can interpret $\tilde{y}_{1t}$ as a complex vector of m_1 transformed predetermined variables and $\tilde{y}_{2t}$ as a complex vector of m_2 transformed non-predetermined variables. Premultiply the system (B.2) by Q and use (B.3)-(B.5) to write it as

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} \mathrm{E}_t\tilde{y}_{1,t+1} \\ \mathrm{E}_t\tilde{y}_{2,t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} \tilde{y}_{1t} \\ \tilde{y}_{2t} \end{bmatrix} + \begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} \mathrm{E}_t\theta_{t+1}, \tag{B.6}$$

where S, T, and Q have been partitioned conformably with $\tilde{y}_{1t}$ and $\tilde{y}_{2t}$.

Consider the lower block of (B.6),

$$S_{22}\,\mathrm{E}_t\tilde{y}_{2,t+1} = T_{22}\,\tilde{y}_{2t} + Q_{21}\mathrm{E}_t\theta_{t+1}. \tag{B.7}$$

Since the diagonal terms of S_{22} and T_{22} (s_{jj} and t_{jj} for $m_1+1 \leq j \leq m_1+m_2$) satisfy $|t_{jj}/s_{jj}| > 1$, the diagonal terms of T_{22} are nonzero, the determinant of T_{22} is nonzero, and T_{22} is invertible. Note that S_{22} may not be invertible. I can then solve for $\tilde{y}_{2t}$ as

$$\begin{aligned} \tilde{y}_{2t} &= J\mathrm{E}_t\tilde{y}_{2,t+1} + K\mathrm{E}_t\theta_{t+1} && \text{(B.8)} \\ &= \sum_{\tau=0}^{\infty} J^{\tau}K\mathrm{E}_t\theta_{t+1+\tau} && \text{(B.9)} \end{aligned}$$

for $t \geq 0$, where the complex matrices J and K ($m_2 \times m_2$ and $m_2 \times m_1$, respectively) are given by

$$\begin{aligned} J &\equiv T_{22}^{-1}S_{22}, && \text{(B.10)} \\ K &\equiv -T_{22}^{-1}Q_{21}. && \text{(B.11)} \end{aligned}$$

Here, I have exploited that the modulus of the diagonal terms of $T_{22}^{-1}S_{22}$ is less than one. I also assume that $\mathrm{E}_t\tilde{y}_{2,t+\tau}$ and $\mathrm{E}_t\theta_{t+\tau}$ are sufficiently bounded. Then $J^{\tau}\mathrm{E}_t\tilde{y}_{2,t+\tau} \to 0$ when $\tau \to \infty$, and

[40] The sorting of the eigenvalues is often done by two programs written by Sims and available at www.princeton.edu/∼sims, Qzdiv and Qzswitch.

the infinite sum on the right side converges. Note that J may not be invertible, since S_{22} may not be invertible.

I have, by (B.5),

$$y_{1t} = Z_{11}\tilde{y}_{1t} + Z_{12}\tilde{y}_{2t}, \tag{B.12}$$

$$y_{2t} = Z_{21}\tilde{y}_{1t} + Z_{22}\tilde{y}_{2t}, \tag{B.13}$$

where

$$Z \equiv \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \tag{B.14}$$

is partitioned conformably with y_{1t} and y_{2t}. Under the assumption of the saddle-point property, Z_{11} is square. I furthermore assume that Z_{11} is *invertible*. Then I can solve for $\tilde{y}_{1t}$ in (B.12),

$$\tilde{y}_{1t} = Z_{11}^{-1}y_{1t} - Z_{11}^{-1}Z_{12}\tilde{y}_{2t}, \tag{B.15}$$

and use this in (B.13) to get

$$y_{2t} = F_1 y_{1t} + H\tilde{y}_{2t}, \tag{B.16}$$

where the real $m_2 \times m_1$ matrix F_1 and the complex $m_2 \times m_2$ matrix H are given by

$$F_1 \equiv Z_{21}Z_{11}^{-1}, \tag{B.17}$$

$$H \equiv Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}. \tag{B.18}$$

I will show below that H is invertible.

By (B.9) and (B.16), I can then write the solution of y_{2t} as

$$y_{2t} = F_1 y_{1t} + Z_t, \tag{B.19}$$

where Z_t is a real exogenous m_2-vector stochastic process (not to be confused with the unitary matrix Z in the Schur decomposition) given by

$$Z_t \equiv H\tilde{y}_{2t} \equiv \sum_{\tau=0}^{\infty} R_\tau \mathrm{E}_t\theta_{t+1+\tau}, \tag{B.20}$$

$$R_\tau \equiv HJ^\tau K \qquad (\tau \geq 0), \tag{B.21}$$

where the matrices R_τ are real.

I note that the complex conjugate transpose of Z, Z', satisfies

$$Z' \equiv \begin{bmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{bmatrix}, \tag{B.22}$$

where the submatrices are given by (B.14). Since $Z'Z = ZZ' = I$, I have

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} Z'_{11} & Z'_{21} \\ Z'_{12} & Z'_{22} \end{bmatrix} = \begin{bmatrix} Z_{11}Z'_{11} + Z_{12}Z'_{12} & Z_{11}Z'_{21} + Z_{12}Z'_{22} \\ Z_{21}Z'_{11} + Z_{22}Z'_{12} & Z_{21}Z'_{21} + Z_{22}Z'_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \tag{B.23}$$

By (B.22), I can write

$$\tilde{y}_{2t} = Z'_{12}y_{1t} + Z'_{22}y_{2t}.$$

Using this in (B.16) gives

$$\begin{aligned} y_{2t} &= F_1 y_{1t} + H(Z'_{12}y_{1t} + Z'_{22}y_{2t}) \\ &= (F_1 + HZ'_{12})y_{1t} + HZ'_{22}y_{2t}. \end{aligned}$$

It follows that

$$F_1 + HZ'_{12} = 0, \tag{B.24}$$

$$HZ'_{22} = I. \tag{B.25}$$

I can also show (B.24) by using (B.23),

$$\begin{aligned} Z_{21}Z_{11}^{-1} + (Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})Z'_{12} &= Z_{21}Z_{11}^{-1} + Z_{22}Z'_{12} - Z_{21}Z_{11}^{-1}Z_{12}Z'_{12} \\ &= Z_{21}Z_{11}^{-1} + Z_{22}Z'_{12} - Z_{21}Z_{11}^{-1}(I - Z_{11}Z'_{11}) \\ &= Z_{21}Z_{11}^{-1} + Z_{22}Z'_{12} - Z_{21}Z_{11}^{-1} + Z_{21}Z'_{11} \\ &= 0. \end{aligned}$$

Similarly, I can show (B.25) by

$$\begin{aligned} (Z_{22} - Z_{21}Z_{11}^{-1}Z_{12})Z'_{22} &= Z_{22}Z'_{22} - Z_{21}Z_{11}^{-1}Z_{12}Z'_{22} \\ &= Z_{22}Z'_{22} - Z_{21}Z_{11}^{-1}(-Z_{11}Z'_{21}) \\ &= Z_{22}Z'_{22} + Z_{21}Z'_{21} \\ &= I. \end{aligned}$$

It follows from (B.25) that H is invertible and that its inverse is given by

$$H^{-1} = Z'_{22}. \tag{B.26}$$

It remains to find a solution for $y_{1,t+1}$. The upper block of (B.6) is

$$S_{11}\mathrm{E}_t\tilde{y}_{1,t+1} + S_{12}\mathrm{E}_t\tilde{y}_{2,t+1} = T_{11}\tilde{y}_{1t} + T_{12}\tilde{y}_{2t} + Q_{11}\mathrm{E}_t\theta_{t+1}.$$

Since the diagonal terms of S_{11} and T_{11} satisfy $|t_{jj}/s_{jj}| < 1$, all diagonal terms of S_{11} must be nonzero, so the determinant of S_{11} is nonzero, and S_{11} is invertible. I can then solve for $\mathrm{E}_t\tilde{y}_{1,t+1}$ as

$$\mathrm{E}_t\tilde{y}_{1,t+1} = S_{11}^{-1}(T_{11}\tilde{y}_{1t} + T_{12}\tilde{y}_{2t}) - S_{11}^{-1}S_{12}\mathrm{E}_t\tilde{y}_{2,t+1} + S_{11}^{-1}Q_{11}\mathrm{E}_t\theta_{t+1}.$$

By (B.12),

$$\begin{aligned}
\mathrm{E}_t y_{1,t+1} &= Z_{11}\mathrm{E}_t\tilde{y}_{1,t+1} + Z_{12}\mathrm{E}_t\tilde{y}_{2,t+1} \\
&= Z_{11}[S_{11}^{-1}(T_{11}\tilde{y}_{1t} + T_{12}\tilde{y}_{2t}) - S_{11}^{-1}S_{12}\mathrm{E}_t\tilde{y}_{2,t+1} + S_{11}^{-1}Q_{11}\mathrm{E}_t\theta_{t+1}] + Z_{12}\mathrm{E}_t\tilde{y}_{2,t+1} \\
&= Z_{11}S_{11}^{-1}T_{11}\tilde{y}_{1t} + Z_{11}S_{11}^{-1}T_{12}\tilde{y}_{2t} + (Z_{12} - Z_{11}S_{11}^{-1}S_{12})\mathrm{E}_t\tilde{y}_{2,t+1} + Z_{11}S_{11}^{-1}Q_{11}\mathrm{E}_t\theta_{t+1} \\
&= Z_{11}S_{11}^{-1}T_{11}(Z_{11}^{-1}y_{1t} - Z_{11}^{-1}Z_{12}\tilde{y}_{2t}) + Z_{11}S_{11}^{-1}T_{12}\tilde{y}_{2t} + (Z_{12} - Z_{11}S_{11}^{-1}S_{12})\mathrm{E}_t\tilde{y}_{2,t+1} \\
&\quad + Z_{11}S_{11}^{-1}Q_{11}\mathrm{E}_t\theta_{t+1} \\
&= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}y_{1t} + Z_{11}S_{11}^{-1}(T_{12} - T_{11}Z_{11}^{-1}Z_{12})\tilde{y}_{2t} \\
&\quad + (Z_{12} - Z_{11}S_{11}^{-1}S_{12})\mathrm{E}_t\tilde{y}_{2,t+1} + Z_{11}S_{11}^{-1}Q_{11}\mathrm{E}_t\theta_{t+1} \\
&= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}y_{1t} + Z_{11}S_{11}^{-1}(T_{12} - T_{11}Z_{11}^{-1}Z_{12})(J\mathrm{E}_t\tilde{y}_{2,t+1} + K\mathrm{E}_t\theta_{t+1}) \\
&\quad + (Z_{12} - Z_{11}S_{11}^{-1}S_{12})\mathrm{E}_t\tilde{y}_{2,t+1} + Z_{11}S_{11}^{-1}Q_{11}\mathrm{E}_t\theta_{t+1} \\
&= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}y_{1t} \\
&\quad + [Z_{11}S_{11}^{-1}(T_{12} - T_{11}Z_{11}^{-1}Z_{12})J + (Z_{12} - Z_{11}S_{11}^{-1}S_{12})]\mathrm{E}_t\tilde{y}_{2,t+1} \\
&\quad + Z_{11}S_{11}^{-1}[Q_{11} + (T_{12} - T_{11}Z_{11}^{-1}Z_{12})K]\mathrm{E}_t\theta_{t+1}, \qquad \text{(B.27)}
\end{aligned}$$

where I have used (B.15) and (B.8).

It follows that I can use (B.27), (B.20), and (B.26) and write the solution as

$$y_{1,t+1} = My_{1t} + N\mathrm{E}_t Z_{t+1} + P\mathrm{E}_t\theta_{t+1} + (\theta_{t+1} - \mathrm{E}_t\theta_{t+1}), \qquad \text{(B.28)}$$

where the real matrices M, N, and P are given by

$$\begin{aligned}
M &\equiv Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}, &\text{(B.29)}\\
N &\equiv [Z_{11}S_{11}^{-1}(T_{12} - T_{11}Z_{11}^{-1}Z_{12})J + (Z_{12} - Z_{11}S_{11}^{-1}S_{12})]Z_{22}', &\text{(B.30)}\\
P &\equiv Z_{11}S_{11}^{-1}[Q_{11} + (T_{12} - T_{11}Z_{11}^{-1}Z_{12})K]. &\text{(B.31)}
\end{aligned}$$

Thus, the solution to the system (B.1) is given by (B.19) and (B.28) for $t \geq 0$. This results in the solution (A.11)-(A.12) above, where the matrix P in (A.12) is the submatrix of the first n_X rows of the matrix P in (B.31) (since $\theta_{t+1} \equiv (z_{t+1}', 0')'$).

C. The model when judgment is a finite-order moving average

When the deviation is a finite-order moving-average process and the dynamics of the deviation and judgment is described by (2.16), the model can be written as

$$\begin{bmatrix} X_{t+1} \\ z^{t+1} \\ Cx_{t+1|t} \end{bmatrix} = \bar{A} \begin{bmatrix} X_t \\ z^t \\ x_t \end{bmatrix} + \bar{B} i_t + \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon^{t+1} \\ 0 \end{bmatrix}, \tag{C.1}$$

where the matrices $\bar{A}$ and $\bar{B}$ are given by

$$\bar{A} \equiv \begin{bmatrix} A_{11} & A_{z12} & A_{12} \\ 0 & A_{z22} & 0 \\ A_{21} & 0 & A_{22} \end{bmatrix}, \qquad \bar{B} \equiv \begin{bmatrix} B_1 \\ 0 \\ B_2 \end{bmatrix},$$

the matrix A_z is partitioned conformably with z_t and z^t as

$$A_z \equiv \begin{bmatrix} 0 & A_{z12} \\ 0 & A_{z22} \end{bmatrix},$$

and $\tilde{\varepsilon}_t \equiv (\varepsilon_t', \varepsilon^{t\prime})'$ is zero-mean and iid. Thus, this results in the standard forward-looking linear-quadratic model, with the predetermined variables being X_t and z^t. The optimal policy projection can then be described as (2.17) and (2.18), where F and M are finite-dimensional matrices. The intertemporal loss for the optimal policy projection can then be written as

$$\frac{1}{2} \begin{bmatrix} X_t \\ z^t \\ \Xi_{t-1,t-1} \end{bmatrix}' V \begin{bmatrix} X_t \\ z^t \\ \Xi_{t-1,t-1} \end{bmatrix},$$

where the matrix V is the solution to the Lyapunov equation,

$$V = \bar{W} + \delta M' V M,$$

the symmetric and positive semidefinite matrix $\bar{W}$ is defined by

$$\bar{W} = \begin{bmatrix} I \;\; 0 \;\; 0 \\ F_x \\ F_i \end{bmatrix}' D' W D \begin{bmatrix} I \;\; 0 \;\; 0 \\ F_x \\ F_i \end{bmatrix},$$

and the matrix F is partitioned conformably with x_t and i_t as

$$F \equiv \begin{bmatrix} F_x \\ F_i \end{bmatrix}.$$

D. The Marcet-Marimon method to solve the linear-quadratic optimization problem with forward-looking variables

Let $\bar{X}_t \equiv (X_t, z^t)$ and write the model (C.1) as

$$\bar{X}_{t+1} = \bar{A}_{11}\bar{X}_t + \bar{A}_{12}x_t + \bar{B}_1 i_t + \tilde{\varepsilon}_{t+1}, \tag{D.1}$$

$$C\mathrm{E}_t x_{t+1} = \bar{A}_{21}\bar{X}_t + \bar{A}_{22}x_t + \bar{B}_2 i_t. \tag{D.2}$$

Write the period loss function as

$$L_t = \frac{1}{2}\begin{bmatrix} \bar{X}_t \\ x_t \\ i_t \end{bmatrix}' W^0 \begin{bmatrix} \bar{X}_t \\ x_t \\ i_t \end{bmatrix}, \tag{D.3}$$

where the symmetric positive semidefinite matrix W^0 is defined by

$$\begin{bmatrix} \bar{X}_t \\ x_t \\ i_t \end{bmatrix}' W^0 \begin{bmatrix} \bar{X}_t \\ x_t \\ i_t \end{bmatrix} \equiv \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}' D'WD \begin{bmatrix} X_t \\ x_t \\ i_t \end{bmatrix}.$$

Consider the problem in period 0,

$$\min_{\{i_t\}_{t\geq 0}} \mathrm{E}_0 \sum_{t=0}^{\infty} \delta^t L_t, \tag{D.4}$$

subject to (D.1), (D.2) and X_0 given. The minimization is taken to be under commitment.

Marcet and Marimon [14] show that this problem can be reformulated as a recursive saddlepoint problem,

$$\max_{\{\gamma_t\}_{t\geq 0}} \min_{\{x_t, i_t\}_{t\geq 0}} \mathrm{E}_0 \sum_{t=0}^{\infty} \delta^t \tilde{L}_t, \tag{D.5}$$

where the modified period loss function satisfies

$$\begin{aligned} \tilde{L}_t &\equiv \tilde{L}(\bar{X}_t, \Xi_{t-1}; x_t, i_t, \gamma_t) \\ &\equiv L_t + L_t^1 \\ &\equiv L_t + \gamma_t'(-\bar{A}_{21}\bar{X}_t - \bar{A}_{22}x_t - \bar{B}_2 i_t) + \frac{1}{\delta}\Xi_{t-1}' C x_t, \end{aligned}$$

and the optimization is subject to (D.1), to

$$\Xi_t = \gamma_t, \tag{D.6}$$

and to X_0 and $\Xi_{-1} = 0$ given. The value function for the saddlepoint problem, starting in any period t, satisfies

$$\tilde{V}(\bar{X}_t, \Xi_{t-1}) \equiv \max_{\gamma_t} \min_{(x_t, i_t)} \{\tilde{L}(\bar{X}_t, \Xi_{t-1}; x_t, i_t, \gamma_t) + \delta \mathrm{E}_t \tilde{V}(\bar{X}_{t+1}, \Xi_t)\},$$

subject to (D.1) and (D.6).

Define

$$\tilde{X}_t \equiv \begin{bmatrix} \bar{X}_t \\ \Xi_{t-1} \end{bmatrix}, \qquad \tilde{\imath}_t \equiv \begin{bmatrix} x_t \\ i_t \\ \gamma_t \end{bmatrix},$$

and define $\bar{W}$, $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ such that

$$\tilde{L}_t \equiv \frac{1}{2} \begin{bmatrix} \tilde{X}_t \\ \tilde{\imath}_t \end{bmatrix}' \bar{W} \begin{bmatrix} \tilde{X}_t \\ \tilde{\imath}_t \end{bmatrix}, \tag{D.7}$$

$$\tilde{X}_{t+1} = \tilde{A}\tilde{X}_t + \tilde{B}\tilde{\imath}_t + \tilde{C}\tilde{\varepsilon}_{t+1}. \tag{D.8}$$

The problem (D.5) subject to (D.8) and given $\tilde{X}_t$ is isomorphic to a standard backward-looking linear-quadratic problem, except being a saddlepoint problem. However, the saddlepoint aspect does not affect the first-order conditions. It is easy to show that the first-order conditions of the saddlepoint problem are identical to those of the original problem, (D.4) subject to (D.1) and (D.2).

The value function for the saddlepoint problem is quadratic,

$$\tilde{V}(\tilde{X}_t) \equiv \frac{1}{2}(\tilde{X}_t'\tilde{V}\tilde{X}_t + \tilde{w}),$$

where $\tilde{V}$ solves the Riccati equation,

$$\tilde{V} = Q + \delta\tilde{A}'\tilde{V}\tilde{A} - (\delta\tilde{B}'\tilde{V}\tilde{A} + N')'(\delta\tilde{B}'\tilde{V}\tilde{B} + R)^{-1}(\delta\tilde{B}'\tilde{V}\tilde{A} + N'),$$

where

$$\bar{W} \equiv \begin{bmatrix} Q & N \\ N' & R \end{bmatrix},$$

is partitioned conformably with $\tilde{X}_t$ and $\tilde{\imath}_t$.

The optimal reaction function for the saddlepoint problem is linear,

$$\tilde{\imath}_t = F\tilde{X}_t \equiv \begin{bmatrix} F_x \\ F_i \\ F_\gamma \end{bmatrix} \tilde{X}_t,$$

where F is partitioned conformably with x_t, i_t, and γ_t and satisfies

$$F \equiv -(\delta\tilde{B}'\tilde{V}\tilde{B} + R)^{-1}(\delta\tilde{B}'\tilde{V}\tilde{A} + N').$$

This reaction function function is the optimal reaction function function for the original problem. Optimization in a timeless perspective in period t corresponds to taking Ξ_{t-1} from the previous period's decision problem as given, also in period 0.

The equilibrium dynamics will be given by

$$
\begin{aligned}
\tilde{X}_{t+1} &= M\tilde{X}_t + \tilde{C}\varepsilon_{t+1}, \\
x_t &= F_x\tilde{X}_t, \\
i_t &= F_i\tilde{X}_t, \\
L_t &= \frac{1}{2}\tilde{X}_t'\tilde{W}\tilde{X}_t,
\end{aligned}
$$

where

$$M \equiv \tilde{A} + \tilde{B}\tilde{F},$$

$$\tilde{W} \equiv \begin{bmatrix} I \ \ 0 \\ F_x \\ F_i \end{bmatrix}' W^0 \begin{bmatrix} I \ \ 0 \\ F_x \\ F_i \end{bmatrix}.$$

The value function for the saddlepoint problem can be decomposed according to

$$\frac{1}{2}(\tilde{X}_t'\tilde{V}\tilde{X}_t + \tilde{w}) \equiv \frac{1}{2}(\tilde{X}_t'V\tilde{X}_t + w) + \frac{1}{2}(\tilde{X}_t'V^1\tilde{X}_t + w^1),$$

where

$$\frac{1}{2}(\tilde{X}_t'V\tilde{X}_t + w) \equiv \mathrm{E}_t\sum_{\tau=0}^{\infty}\delta^{\tau-t}\frac{1}{2}\tilde{X}_{t+\tau}'\tilde{W}\tilde{X}_{t+\tau},$$

is the value function for the original problem starting in period t with $\tilde{X}_t \equiv (X_t', \Xi_{t-1}')'$ given. The matrix V will satisfy the Lyapunov equation,

$$V = \tilde{W} + \delta M'VM,$$

and, when $\delta < 1$, the constant w will satisfy

$$w = \frac{\delta}{1-\delta}\mathrm{tr}(\tilde{C}'V\tilde{C}\Sigma_{\tilde{\varepsilon}\tilde{\varepsilon}}),$$

where $\Sigma_{\tilde{\varepsilon}\tilde{\varepsilon}}$ is the covariance matrix for $\tilde{\varepsilon}_t$.

E. An alternative finite-horizon numerical procedure for forward-looking models

In the finite-horizon model in section 3.1, there is an obvious alternative numerical procedure that will provide a projection arbitrarily close to the optimal policy projection without requiring such a long horizon that $X_{t+T,t}$ and $\Xi_{t+T-1,t}$ are close to their steady-state levels. It requires iterations, though.

Assume that iteration $j-1$ has resulted in $\Xi_{t+T-1,t}^{(j-1)}$. Start iteration j by using (2.17) and (2.18) to replace (3.3) by

$$x_{t+T+1,t} = F_x M_1 \begin{bmatrix} X_{t+T,t} \\ \Xi_{t+T-1,t}^{(j-1)} \end{bmatrix},$$

where the matrices F_1 and M_1 are defined by

$$F \begin{bmatrix} X_t \\ 0 \\ \Xi_{t-1} \end{bmatrix} \equiv F_1 \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix}, \qquad M \begin{bmatrix} X_t \\ 0 \\ \Xi_{t-1} \end{bmatrix} \equiv M_1 \begin{bmatrix} X_t \\ \Xi_{t-1} \end{bmatrix},$$

and F_1 is partitioned conformably with x_t and i_t as

$$F_1 \equiv \begin{bmatrix} F_x \\ F_i \end{bmatrix}.$$

Consequently, replace (3.4) by

$$-A_{21}X_{t+T,t} - A_{22}x_{t+T,t} - B_2 i_{t+T,t} + CF_x M_1 \begin{bmatrix} X_{t+T,t} \\ \Xi_{t+T-1,t}^{(j-1)} \end{bmatrix} = 0. \qquad \text{(E.1)}$$

Use (3.1), (3.2), and (E.1) to construct G and g^t (the left submatrix of the matrix CF_xM_1 will enter the last block of G and the product of the right submatrix and $\Xi_{t+T-1,t}^{(j-1)}$ will enter the last block of g^t). Furthermore, add the term (3.7) with $\Xi_{t+T-1,t} = \Xi_{t+T-1,t}^{(j-1)}$ to the loss function (that is, modify the diagonal block of Ω that corresponds to $X_{t+T,t}$ and add a linear term that corresponds to the cross products of $X_{t+T,t}$ and $\Xi_{t+T-1,t}^{(j-1)}$). Find the optimal policy projection $\hat{s}^{t(j)}$ and Lagrange multiplier $\Lambda^{t(j)}$ via the analogue of (3.12). This ends iteration j and results in $\Xi_{t+T-1,t}^{(j)}$. Continue until $\Xi_{t+T-1,t}^{(j)}$ has converged.

Obviously this alternative procedure does not require that $X_{t+T,t}$ and $\Xi_{t+T-1,t}$ are close to their steady-state levels. Which procedure is fastest will depend on the number of variables in the problem and the rate of convergence towards the steady state of the optimal policy projection.

F. The feasible set of projections of the states of the economy, the feasible set of projections of the target variables, and the optimal targeting rule

In the finite-horizon projection model in section 3.1, the feasible set of projections in period t of the states of the economy, $\mathcal{S}_t$, is the set of projections s^t that satisfy (3.5), repeated here as

$$Gs^t = g^t. \qquad \text{(F.1)}$$

That is, $\mathcal{S}_t$ is the set of solutions to (F.1) for given G and g^t. Define $n \equiv (T+1)(n_X + n_x + n_i)$, $m \equiv (T+1)(n_X + n_x) < n$, and $p \equiv (T+1)n_i \equiv n - m$. Note that G is $m \times n$, s^t is $n \times 1$, and g^t is $m \times 1$. Assume that G is of rank m.

Since G is of rank m, the set of solutions to (F.1) is a linear manifold of R^n of dimension $p \equiv n - m$.[41] It can be written as the set of projections s^t that satisfy

$$s^t = G^+ g^t + (I - G^+ G)\xi \tag{F.2}$$

for any $\xi \in R^n$ (see Harville [A2, chapters 11 and 20]). Here, the $n \times m$ matrix G^+ is the Moore-Penrose inverse of G. When G is $m \times n$ and of rank m, the Moore-Penrose inverse is given by

$$G^+ = G'(GG')^{-1}$$

(note that GG' is $m \times m$, of rank m, and hence invertible). Then, $G^+G = G'(GG')^{-1}G$ is a projection matrix that projects vectors in R^n on the m-dimensional column space of the $n \times m$ matrix G', the transpose of G.[42] Denote the column space of G' by $\mathcal{C}(G')$. For any ξ in R^n, the vector $G^+G\xi$ lies in $\mathcal{C}(G')$. Then $I - G^+G$ is a projection matrix that projects vectors in R^n *off* the column space of G', that is, on the p-dimensional subspace of R^n orthogonal to $\mathcal{C}(G')$, the orthogonal complement of $\mathcal{C}(G')$ (relative to R^n), denoted $\mathcal{C}^\perp(G')$. Hence, the solution set $\mathcal{S}_t$ consist of $\mathcal{C}^\perp(G')$ shifted away from the origin by the vector G^+g^t,

$$\mathcal{S}_t = \{G^+g^t\} + \mathcal{C}^\perp(G').$$

Furthermore, the vector G^+g^t is the s^t of minimum norm that satisfies (F.1). Then, G^+g^t is orthogonal to the solution set $\mathcal{S}_t$ and lies in the column space of G, $\mathcal{C}(G')$.[43]

Figure F.1 provides an illustration of the above, when $n = 2$ and $m = p = 1$. The linear manifold $\mathcal{S}_t$, the set of feasible projections of the states of the economy, s^t, is shown as the negatively sloped line through the point $s^t = G^+g^t$. The column space $\mathcal{C}(G')$ is the positively sloped line through the origin. The linear manifold $\mathcal{S}_t$ is orthogonal to the column space. The orthogonal complement of the column space, $\mathcal{C}^\perp(G')$, is the negatively sloped line through the origin. The linear manifold is the orthogonal complement shifted away from the origin to the point G^+g^t. Furthermore, the point G^+g^t is the point in the linear manifold with the shortest distance to the origin.

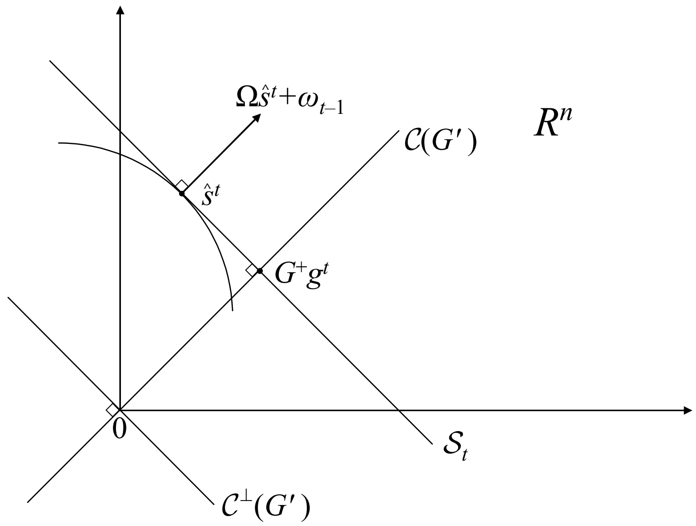
Let $G^\perp$ denote a $p \times n$ matrix with p linearly independent rows, each of which is orthogonal to the m rows of G. Then $\mathcal{C}^\perp(G') = \mathcal{C}(G^{\perp\prime})$, where the latter expression denotes the column space of

[41] Let V be a linear space. A subset S of V is a linear manifold of V (also called a linear variety of V), if there is a v in V such that the set $S - \{v\} \equiv \{s - v | s \in S\}$ is a subspace of V. The dimension of S is the dimension of $S - \{v\}$. Hence, a linear manifold is a subspace that has possibly been shifted away from the origin (in the above case by the vector v).

[42] In this section, the word "projection" is used not only to refer to mean forecasts but also, depending on the context, to refer to mathematical projections in linear space.

[43] A vector is orthogonal to a linear manifold if it is orthogonal to the corresponding subspace.

Figure F.1: The set of feasible projections of the state of the economy, $\mathcal{S}_t$



$G^{\perp\prime}$, and $\mathcal{S}_t$ can be written as the set of projections s^t that satisfy

$$s^t = G^+ g^t + G^{\perp\prime}\xi$$

for any $\xi \in R^n$.

The projection of the target variables, Y^t, is a linear function of the projection of the states of the economy according to (3.6), repeated here as

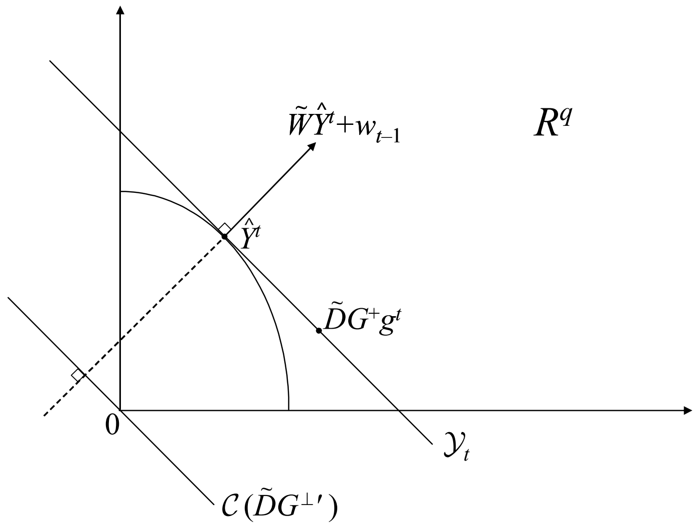
$$Y^t = \tilde{D}s^t. \tag{F.3}$$

Let $q \equiv (T+1)n_Y \leq n$, note that Y^t is $q \times 1$ and $\tilde{D}$ is $q \times n$, and take $\tilde{D}$ to be of rank q. It follows that the set of feasible projections of the target variables, $\mathcal{Y}_t$, consists of the set of projections Y^t that satisfy

$$Y^t = \tilde{D}G^+ g^t + \tilde{D}G^{\perp\prime}\xi$$

for any ξ in R^n. This is a linear manifold of R^q of dimension at most $\min(p, q)$. If I take as the normal case that the number of target variables is at least as large as the number of instruments, $n_Y \geq n_i$ (typically, there are at least two target variables, inflation and the output gap, but only one instrument, the instrument rate), I have $q \geq p$, and the set of feasible projections of the target variables, $\mathcal{Y}_t$, is a linear manifold of R^q of dimension at most $p \leq q$. The matrix $\tilde{D}$ simply maps the p-dimensional linear manifold $\mathcal{S}_t$ of R^n into the at most p-dimensional linear manifold $\mathcal{Y}_t$ of R^q.

Figure F.2: The set of feasible projections of the target variables, $\mathcal{Y}_t$



It follows that $\mathcal{Y}_t$ is the at most p-dimensional column space $\mathcal{C}(\tilde{D}G^{\perp\prime})$ in R^q shifted away from the origin by the vector $\tilde{D}G^+g^t$,

$$\mathcal{Y}_t = \{\tilde{D}G^+g^t\} + \mathcal{C}(\tilde{D}G^{\perp\prime}).$$

Figure F.2 provides an illustration of the above, when $q = 2$ and $p = 1$. The linear manifold $\mathcal{Y}_t$, the set of feasible projections of the target variables, Y^t, is shown as the negatively sloped line through the point $Y^t = \tilde{D}G^+g^t$. The column space of the matrix $\tilde{D}G^{\perp\prime}$, $\mathcal{C}(\tilde{D}G^{\perp\prime})$, is shown as the negative sloped line through the origin. The linear manifold $\mathcal{Y}_t$ is this column space shifted away from the origin to the point $\tilde{D}G^+g^t$.

F.1. An optimal targeting rule for the forward-looking model

Consider the first-order condition for optimal policy under commitment in a timeless perspective in the forward-looking model, (3.10), rewritten here as

$$\Omega s^t + \omega_{t-1} + G'\Lambda^t = 0 \tag{F.4}$$

The *optimal targeting rule* is the first-order condition in terms of Y^t when the Lagrange multiplier has been eliminated.

Let me interpret the first-order condition in terms of s^t, eliminate the Lagrange multiplier, and interpret the resulting targeting rule. Note that Ω is $n \times n$, s^t and ω_{t-1} are $n \times 1$, G' is $n \times m$ and of rank m, and Λ^t is $m \times 1$.

Write the first-order condition as

$$\Omega s^t + \omega_{t-1} = G'(-\Lambda^t). \tag{F.5}$$

The term $\Omega s^t + \omega_{t-1}$ on the left side is the gradient of the loss function with respect to s^t, a vector in R^n. The condition (F.5) can be interpreted as stating that the gradient of the loss function is an element of the m-dimensional column space of the $n \times m$ matrix G', $\mathcal{C}(G')$, with $-\Lambda^t$ providing the coefficients of the corresponding linear combination of the column vectors of G'. This is equivalent to the tangency of the loss function's iso-loss surface in R^n with the feasible set of projections, $\mathcal{S}_t$. The gradient of the loss function is orthogonal to the iso-loss surface. Tangency of the iso-loss surface with $\mathcal{S}_t$ is then equivalent to the gradient being orthogonal to $\mathcal{S}_t$. The subspace orthogonal to $\mathcal{S}_t$ is $\mathcal{C}(G')$, as noted above.

This is illustrated in figure F.1 when $n = 2$ and $m = p = 1$. The curve shows part of the iso-loss surface of the loss function that is tangential to the linear manifold $\mathcal{S}_t$. The tangency occurs at the optimal policy projection, $\hat{s}^t$. The gradient of the loss function at that point, $\Omega \hat{s}^t + \omega_{t-1}$, is shown as the vector pointing northeast from that point. Tangency between the iso-loss surface and the linear manifold is equivalent to the gradient being orthogonal to the linear manifold, or the gradient being an element in the column space, $\mathcal{C}(G')$.

In order to eliminate the Lagrange multipliers, premultiply (F.5) by G,[44]

$$G(\Omega s^t + \omega_{t-1}) = GG'(-\Lambda^t). \tag{F.6}$$

Exploit that GG' is $m \times m$, of rank m, and hence invertible, and solve for $-\Lambda^t$,

$$-\Lambda^t = (GG')^{-1}G(\Omega s^t + \omega_{t-1}). \tag{F.7}$$

(The matrix $(GG')^{-1}G$ is actually the Moore-Penrose inverse of G', G'^{+}, where G' is $n \times m$ with rank m.) Substitution of Λ^t in (F.4) gives

$$M(\Omega s^t + \omega_{t-1}) = 0, \tag{F.8}$$

where M is the $n \times n$ matrix (not to be confused with the matrix denoted M in other sections of this paper)

$$M \equiv I - G'(GG')^{-1}G = I - G^{+}G.$$

[44] One might ask why multipliying with the matrix G with rank $m < n$ rather than a matrix with full rank n does not loose any information of (F.5). More formally, let $G^{\perp}$ be a $p \times n$ matrix whose p rows are linearly independent and orthogonal to the m rows of G. That is, the column space of $G^{\perp\prime}$ is the space in R^n orthogonal to the column space of G'. Then the $n \times n$ matrix $\begin{bmatrix} G \\ G^{\perp} \end{bmatrix}$ is of full rank. Multiplying (F.5) by this matrix leads to the m equations of (F.6) and p additional trivial equations of zero equals zero, since we know that the left and right sides of (F.5) lie in the column space of G'.

As noted above, M is the projection matrix that projects vectors in R^n on the p-dimensional orthogonal complement of the column space of G', $\mathcal{C}^\perp(G')$. Hence, (F.8) states that the projection on $\mathcal{C}^\perp(G')$ of the gradient of the loss function is zero. Of course, this follows directly from the observation above that the gradient lies in $\mathcal{C}(G')$.

In any case, the optimal targeting rule in terms of s^t is equivalent to the statement that the gradient is orthogonal to the feasible set of projections of the states of the economy, $\mathcal{S}_t$, which can be expressed algebraically as (F.8).

However, (F.8) involves n equations, but only p independent equations. It is hence desirable to condense (F.8) to only p equations. The projection matrix M is a symmetric idempotent matrix of rank p. Then its spectrum consists of p eigenvalues equal to one and m eigenvalues equal to zero, and it can be decomposed as

$$M = Q \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} Q' \equiv \begin{bmatrix} Q_p & Q_m \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_p' \\ Q_m' \end{bmatrix} \equiv Q_p Q_p'.$$

Here Q is the orthonormal $n \times n$ matrix whose columns are the eigenvectors of M, I_p is the $p \times p$ identity matrix, and Q_p is the $n \times p$ matrix whose columns are the p eigenvectors corresponding to the p nonzero eigenvalues. Then, pre-multiplying (F.8) by Q' gives the p nontrivial equations,

$$Q_p'(\Omega s^t + \omega_{t-1}) = 0, \tag{F.9}$$

and m trivial equations of zero equals zero.

Furthermore, (F.9) is expressed in terms of the projection of the states of the economy, s^t. In order to express it in terms of the projection of the target variables, Y^t, note that, by the definition of Ω for the forward-looking model in section 3.1,

$$\Omega s^t \equiv \tilde{D}'\tilde{W}\tilde{D}s^t \equiv \tilde{D}'\tilde{W}Y^t,$$

where $\tilde{W}$ is a symmetric positive semidefinite block-diagonal $(T+1)n_Y$ matrix with the $(\tau+1)$-th diagonal block being $\delta^\tau W$ for $0 \leq \tau \leq T$. Hence, I can write (F.9) as involving only the target variables and, through the vector ω_{t-1}, the Lagrange multiplier $\Xi_{t-1,t-1}$ from the optimization in period $t-1$,

$$Q_p'[\tilde{D}'\tilde{W}Y^t + \omega_{t-1}] = 0. \tag{F.10}$$

This is one concise form of the targeting rule. The history-dependence of the optimal policy under commitment in a timeless perspective enters via $\Xi_{t-1,t-1}$.

By combining (F.9) with (3.5), I get

$$\begin{bmatrix} G \\ Q_p'\Omega \end{bmatrix} s^t = \begin{bmatrix} g^t \\ -Q_p'\omega_{t-1} \end{bmatrix},$$

and

$$\hat{s}^t = \begin{bmatrix} G \\ Q_p'\Omega \end{bmatrix}^{-1} \begin{bmatrix} g^t \\ -Q_p'\omega_{t-1} \end{bmatrix} \equiv H \begin{bmatrix} X_t \\ \Xi_{t-1,t-1} \\ z^t \end{bmatrix}, \tag{F.11}$$

$$\hat{Y}^t = \tilde{D}\hat{s}^t = \tilde{D}H \begin{bmatrix} X_t \\ \Xi_{t-1,t-1} \\ z^t \end{bmatrix}.$$

From (F.7) and (F.11), I can extract

$$\Xi_{t,t} = H_\Xi \begin{bmatrix} X_t \\ \Xi_{t-1,t-1} \\ z^t \end{bmatrix},$$

to be used in the intertemporal loss function for the decision problem in period $t+1$.

If the forward-looking variables, x_t, are target variables—elements of Y_t—the intertemporal loss function with the added term can be written

$$\frac{1}{2}Y^{t\prime}\tilde{W}Y^t + w_{t-1}'Y^t,$$

where w_{t-1} is a q-vector whose only nonzero elements contain the vector $(\Xi_{t-1,t-1}\frac{1}{\delta}C)'$ such that $w_{t-1}'Y^t \equiv \Xi_{t-1,t-1}\frac{1}{\delta}Cx_{t,t}$. Then, the optimal targeting rule can be expressed as the gradient, $\tilde{W}Y^t + w_{t-1}$, being orthogonal to the linear manifold $\mathcal{Y}_t$. Suppose $\mathcal{Y}_t$ is of dimension p, and let $F \equiv \tilde{D}G^{\perp\prime}$ (not to be confused with the matrix denoted F in other sections of the paper). The projection matrix that projects vectors in R^q on the p-dimensional subspace $\mathcal{Y}_t - \{\tilde{D}G^+g^t\}$ is then $F(F'F)^{-1}F'$, so the condition that the gradient is orthogonal to the linear manifold $\mathcal{Y}^t$ can be written as the p equations.

$$F(F'F)^{-1}F'\tilde{W}(\Omega Y^t + w_{t-1}) = 0.$$

This is the optimal targeting rule for this case.

This case is illustrated in figure F.2. The curve in the figure shows a part of the iso-loss surface of the loss function that is tangential to the linear manifold $\mathcal{Y}_t$. The tangency point is the optimal policy projection of the target variables, $\hat{Y}^t$. The gradient of the loss function at that point, $\tilde{W}\hat{Y}^t + w_{t-1}$, is shown as the vector at that point that points northeast. It is orthogonal to the linear manifold.

Svensson [25] interprets optimal targeting rules in terms of the equality between the marginal rates of transformation and marginal rates of substitution between the target variables. A vector of marginal rates of transformation between the target variables is a vector in the column space $\mathcal{C}(\tilde{D}G^{\perp\prime})$, the subspace associated with $\mathcal{Y}_t$. A vector of marginal rates of substitution between the target variables is a vector in the tangent space of the intertemporal loss function, the subspace orthogonal to the gradient of the loss function. Equality between the marginal rates of transformation and substitution is equivalent to the gradient being orthogonal to $\mathcal{Y}_t$, that is, the iso-loss surface being tangential to $\mathcal{Y}_t$.

G. An optimal restricted instrument rule

Add to the model (2.1) an explicit instrument rules of the form

$$i_t = f_X X_t, \tag{G.1}$$

where the $n_i \times n_X$ matrix f_X is restricted to be an element $f_X \in \mathcal{F}$ of a given class $\mathcal{F}$ of instrument rules. Assume that the deviation z_t is an iid zero-mean process with variance-covariance matrix Σ. Let the loss function in period t be

$$\lim_{\delta \to 1} \mathrm{E}_t \sum_{\tau=0}^{\infty} (1-\delta)\delta^\tau L_{t+\tau} = \mathrm{E}[L_t],$$

where L_t is given by (2.7). By appendix A, for a given instrument rule f_X, the conditional loss in period t is, for a given δ $(0 < \delta < 1)$, given by

$$\mathrm{E}_t \sum_{\tau=0}^{\infty} (1-\delta)\delta^\tau L_{t+\tau} = \frac{1}{2}\{(1-\delta)X_t' V(f_X,\delta) X_t + \delta \mathrm{trace}[V(f_X,\delta)\Sigma]\},$$

where $V(f_X, \delta)$ is a symmetric positive semidefinite $n_X \times n_X$ matrix that depends on A, B, C, D, W, f_X, and δ. It follows that

$$\mathrm{E}[L_t] = \frac{1}{2}\mathrm{trace}[V(f_X,1)\Sigma].$$

The optimal restricted instrument rule, $\hat{f}_X$, is then given by

$$\hat{f}_X = \arg\min_{f_X \in \mathcal{F}} \frac{1}{2}\mathrm{trace}[V(f_X,1)\Sigma].$$

It depends on the class $\mathcal{F}$ and the variance-covariance matrix Σ, in addition to A, B, C, D, and W.

Note that there is little point in considering implicit instrument rules here,

$$i_t = f_X X_t + f_x x_t. \tag{G.2}$$

For any such implicit instrument rule $f \equiv [f_X \; f_x]$ for which a unique equilibrium exists,

$$x_t = g(f)X_t,$$

where the matrix $g(f)$ depends on f. Then,

$$i_t = [f_X + f_x g(f)]X_t \equiv \tilde{f}_X(f)X_t.$$

That is, for each implicit instrument rule f for which there is a unique equilibrium, there is a unique explicit instrument rule $\tilde{f}_X(f)$ consistent with that equilibrium. Furthermore, for any explicit instrument rule f_X in (G.1) for which there is a unique equilibrium, there is a *continuum* of implicit instrument rules consistent with that equilibrium. For any given instrument rule f_X for which there exists a unique equilibrium,$x_t = g(f_X)X_t$, where the matrix $g(f_X)$ depends on f_X. For any arbitrary $n_i \times n_x$ matrix f_x, I can then write

$$i_t = f_X X_t + f_x[x_t - g(f_X)X_t] = [f_X - f_x g(f_X)]X_t + f_x x_t \equiv \tilde{f}_X(f_X, f_x)X_t + f_x x_t.$$

The only reason for considering implicit instrument rules rather than an explicit instrument rule in this context (when the deviation is an iid zero-mean shock) is when an explicit instrument rule has a determinacy problem—multiple equilibria—in which case one may be able to find a corresponding implicit instrument rule for which there is a unique equilibrium. Svensson and Woodford [30] examine such issues further.

H. An empirical backward-looking model

The two equations of the model of Rudebusch and Svensson [18] are

$$\pi_{t+1} = \alpha_{\pi 1}\pi_t + \alpha_{\pi 2}\pi_{t-1} + \alpha_{\pi 3}\pi_{t-2} + \alpha_{\pi 4}\pi_{t-3} + \alpha_y y_t + z_{\pi,t+1} \tag{H.1}$$

$$y_{t+1} = \beta_{y1}y_t + \beta_{y2}y_{t-1} - \beta_r\left(\frac{1}{4}\Sigma_{j=0}^{3} i_{t-j} - \frac{1}{4}\Sigma_{j=0}^{3}\pi_{t-j}\right) + z_{y,t+1}, \tag{H.2}$$

where π_t is quarterly inflation in the GDP chain-weighted price index (p_t) in percentage points at an annual rate, i.e., $400(\ln p - \ln p_{t-1})$; i_t is the quarterly average federal funds rate in percentage points at an annual rate; y_t is the relative gap between actual real GDP (q_t) and potential GDP (q_t^*) in percentage points, i.e., $100(q_t - q_t^*)/q_t^*$. These five variables were de-meaned prior to estimation, so no constants appear in the equations.

The estimated parameters, using the sample period 1961:1 to 1996:2, are shown in table H.1.

Table H.1

$\alpha_{\pi 1}$	$\alpha_{\pi 2}$	$\alpha_{\pi 3}$	$\alpha_{\pi 4}$	α_y	β_{y1}	β_{y2}	β_r
0.70	−0.10	0.28	0.12	0.14	1.16	−0.25	0.10
(0.08)	(0.10)	(0.10)	(0.08)	(0.03)	(0.08)	(0.08)	(0.03)

The hypothesis that the sum of the lag coefficients of inflation equals one has a p-value of .16, so this restriction was imposed in the estimation.[45]

The state-space form can be written

$$\begin{bmatrix} \pi_{t+1} \\ \pi_t \\ \pi_{t-1} \\ \pi_{t-2} \\ y_{t+1} \\ y_t \\ i_t \\ i_{t-1} \\ i_{t-2} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^4 \alpha_{\pi j} e_j + \alpha_y e_5 \\ e_1 \\ e_2 \\ e_3 \\ \beta_r e_{1:4} + \beta_{y1} e_5 + \beta_{y2} e_6 - \beta_r e_{7:9} \\ e_5 \\ e_0 \\ e_7 \\ e_8 \end{bmatrix} \begin{bmatrix} \pi_t \\ \pi_{t-1} \\ \pi_{t-2} \\ \pi_{t-3} \\ y_t \\ y_{t-1} \\ i_{t-1} \\ i_{t-2} \\ i_{t-3} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\beta_r}{4} \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} i_t + \begin{bmatrix} z_{\pi,t+1} \\ 0 \\ 0 \\ 0 \\ z_{y,t+1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where e_j $(j = 0, 1, ..., 9)$ denotes a 1×9 row vector, for $j = 0$ with all elements equal to zero, for $j = 1, ..., 9$ with element j equal to unity and all other elements equal to zero; and where $e_{j:k}$ $(j < k)$ denotes a 1×9 row vector with elements $j, j+1, ..., k$ equal to $\frac{1}{4}$ and all other elements equal to zero. The predetermined variables are π_t, π_{t-1}, π_{t-2}, π_{t-3}, y_t, y_{t-1}, i_{t-1}, i_{t-2}, i_{t-2}, and i_{t-3}. There are no forward-looking variables.

For a loss function (5.3) with $\delta = 1$, $\lambda = 1$, and $\nu = 0.2$, and the case where z_t is an iid zero-mean shock; the optimal reaction function (2.21) is (the coefficients are rounded to two decimal points),

$$i_t = 1.22\,\pi_t + 0.43\,\pi_{t-1} + 0.53\,\pi_{t-2} + 0.18\,\pi_{t-3} + 1.93\,y_t - 0.49\,y_{t-1} + 0.36\,i_{t-1} - 0.09\,i_{t-2} - 0.05\,i_{t-3}.$$

I. An empirical forward-looking model

An empirical New Keynesian model estimated by Lindé [13] is

$$\begin{aligned} \pi_t &= \omega_f \pi_{t+1|t} + (1-\omega_f)\pi_{t-1} + \gamma y_t + z_{\pi t}, \\ y_t &= \beta_f y_{t+1|t} + (1-\beta_f)(\beta_{y1} y_{t-1} + \beta_{y2} y_{t-2} + \beta_{y3} y_{t-3} + \beta_{y4} y_{t-4}) - \beta_r (i_t - \pi_{t+1|t}) + z_{yt}, \end{aligned}$$

where the restriction $\sum_{j=1}^4 \beta_{yj} = 1$ is imposed. The estimated coefficients are (Table 6a in Lindé [13], non-farm business output) are shown in table I.1.

[45] This p-value was obtained by simulating the above inflation equation 1000 times and ranking the sum of coefficients from the unrestricted Phillips curve estimated from the actual data (i.e., 0.969) in the set of unrestricted sums estimated from the simulated data. This is in the spirit of Rudebusch [A5]. For comparison, the simple t-test gives a p-value of 0.42.

Table I.1

ω_f	γ	β_f	β_r	β_{y1}	β_{y2}	β_{y3}
0.457	0.048	0.425	0.156	1.310	− 0.229	− 0.011
(0.065)	(0.007)	(0.027)	(0.016)	(0.174)	(0.279)	(0.037)

For simplicity, I set $\beta_{y1} = 1$, $\beta_{y2} = \beta_{y3} = \beta_{y4} = 0$. Then the state-space form can be written as

$$\begin{bmatrix} \pi_t \\ y_t \\ i_t \\ z_{\pi,t+1} \\ z_{y,t+1} \\ \omega_f \pi_{t+1|t} \\ \beta_r \pi_{t+1|t} + \beta_f y_{t+1|t} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(1-\omega_f) & 0 & 0 & -1 & 0 & 1 & -\gamma \\ 0 & -(1-\beta_f) & 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \pi_{t-1} \\ y_{t-1} \\ i_{t-1} \\ z_{\pi t} \\ z_{yt} \\ \pi_t \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \beta_r \end{bmatrix} i_t + \begin{bmatrix} 0 \\ 0 \\ 0 \\ z_{\pi,t+1} \\ z_{y,t+1} \\ 0 \\ 0 \end{bmatrix}.$$

The predetermined variables are π_{t-1}, y_{t-1}, i_{t-1}, $z_{\pi t}$, and z_{yt}, and the forward-looking variables are π_t and y_t.

For a loss function (5.3) with $\delta = 1$, $\lambda = 1$, and $\nu = 0.2$, and the case where z_t is an iid zero-mean shock; the optimal reaction function (2.21) is (the coefficients are rounded to two decimal points),

$$i_t = 0.58\,\pi_{t-1} + 0.80\,y_{t-1} + 0.41\,i_{t-1} + 1.06\,z_{\pi t} + 1.38\,z_{yt} + 0.02\,\Xi_{\pi,t-1,t-1} + 0.20\,\Xi_{y,t-1,t-1},$$

where $\Xi_{\pi,t-1,t-1}$ and $\Xi_{y,t-1,t-1}$ are the Lagrange multipliers for the two equations for the forward-looking variables in the decision problem in period $t-1$.

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