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Estimating the Term Structure of Interest Rates for Monetary Policy Analysis

*Magnus Dahlquist and Lars E. O. Svensson**

Stockholm University, S-106 91 Stockholm, Sweden

Abstract

We compare estimation of spot and implied forward interest rates from Swedish Treasury bill and Government bond yields with two functional forms, the simple Nelson & Siegel (NS) and the complex Longstaff & Schwartz (LS). Monetary policy rather than financial analysis is in focus, which affects the evaluation criteria. NS is easier to use and has better convergence properties. LS is more flexible. For the data used, estimates using NS and LS are close, with at most only marginally better fit for LS. The fit of NS seems satisfactory for monetary policy purposes.

I. Introduction

The term structure of interest rates, that is, how interest rates depend on the time to maturity, receives considerable attention in both financial and economic analysis. Estimated spot interest rates (zero-coupon rates) for different maturities and associated implied forward interest rates have long been standard tools for financial analysis in financial markets, for instance in the pricing of financial instruments. The term structure of interest rates, in the form of the yield curve (that is, the yield to maturity on coupon bonds as a function of their time to maturity) and the yield spread between long and short interest rates (that is, the slope of the yield curve), is also a traditional indicator for monetary policy. Long interest rates are usually considered to vary with long-run inflation expectations, and the spread between long and short interest rates is sometimes interpreted as indicating how expansionary or contractionary current monetary policy is.

The recent move to flexible exchange rates in Europe is likely to increase the role of indicators in monetary policy. As a complement or even an

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alternative to the standard yield curve and the yield spread, implied forward interest rates have recently begun to emerge as one of the monetary policy indicators used by several central banks, for instance the Bank of England, Federal Reserve Board and Sveriges Riksbank. Forward rates present the information in the term structure of interest rates in a way that is more easily interpreted for monetary policy purposes, and can under appropriate assumptions be used to infer market expectations of future interest rates, inflation rates and currency depreciation rates more directly than the standard yield curve; see Bank of England (1993) and Svensson (1994b). The standard yield curve of yields to maturity on Treasury bills and Government bonds plotted against the time to maturity is unfortunately not an unambiguous representation of the term structure of interest rates, since almost all bonds are coupon bonds, and yields to maturity on coupon bonds depend on the coupon, the "coupon effect." The term structure therefore needs to be expressed in some standardized way in order to be unambiguous.

Two standardized ways to express the term structure occur in the literature, namely to report a par yield curve consisting of yields to maturity on par bonds (bonds that trade at par and have coupons that equal the yield to maturity) or to report a spot rate curve consisting of yields to maturity on zero-coupon bonds. In practice, few bonds trade at par, and there are few or no zero-coupon bonds available beyond 12 months' maturity. Either way of expressing the term structure then requires estimation of the term structure from yields to maturity on non-par coupon bonds. Even if there were a fair number of par bonds or zero-coupon bonds available, it would still be advantageous to use the information in non-par coupon bond yields as well. This can be done by estimating hypothetical zero-coupon prices from available coupon bonds for longer maturities and zero-coupon bonds (Treasury bills) for shorter maturities. Spot rates and implicit forward rates are then calculated from these. A number of different estimation methods are available.

We follow McCulloch (1971, 1975) in estimating a discount function by fitting model prices to observed bond prices (taking into account coupon payments). However, instead of using the so-called spline technique, which has some disadvantages discussed in Section II, we employ two other functional forms. We use one very simple form suggested by Nelson and Siegel (1987) (NS), and one very complex form derived in a theoretical model by Longstaff and Schwartz (1992) (LS). These functional forms are not nested.

The purpose of the paper is hence to compare the performance of the NS and LS functional forms in estimating spot and forward rates on Swedish term structure data in order to judge which method is more appropriate for monetary policy analysis.

Thus monetary policy analysis rather than financial analysis is in focus, for instance, as concerns the use of forward rates as indicators rather than pricing financial instruments for arbitrage decisions. This has some consequence for the criteria according to which the performance of the two functional forms is evaluated. *First*, somewhat less precision is required for monetary policy purposes than for financial analysis. Yield errors of a few tens of basis points are acceptable for monetary policy analysis, but hardly acceptable for arbitrage decisions. *Second*, from an economics point of view, zero-coupon prices can be interpreted as intertemporal marginal rates of substitution. It seems reasonable to postulate that marginal rates of substitution are rather smooth, in which case it follows that estimates of spot and forward rates should be rather smooth. Smoothness reduces precision but, as mentioned, the demand for precision is less in monetary policy analysis than in financial analysis. Increased demand for precision in financial analysis tends to result in jagged spot rates and volatile forward rates. *Third*, since forward rates can be interpreted as indicating expectations of future interest rates, which in turn depend on expectations of future real interest rates and future inflation rates, it seems reasonable to restrict forward rates for settlements very far into the future to be constant. This is because it seems unlikely that market agents have information that allows them to have different expectations for, say, 25 or 30 years into the future. *Fourth*, the demands on robustness of estimates are probably higher for monetary policy analysis than for financial analysis. The estimates in policy analysis should allow comparisons over time and across countries, with different sets of bonds and Treasury bills, and be less sensitive to missing observations and the number of bonds and bills used in the estimation. In practice, our criteria of evaluation boil down to comparing measures of fit and convergence properties for NS and LS.

The LS functional form is derived in a theoretical model. We want to emphasize that we do not attempt to test the theoretical model, for instance whether the restrictions it imposes are empirically fulfilled or not. We simply use the functional form to fit it to the data, without testing whether the theoretical restrictions are fulfilled.[1]

Section II provides definitions and describes the method of estimating spot and forward rates in general terms. The NS and LS functional forms

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[1] For instance, we deviate from the theory by estimating parameters separately for each trade date, whereas according to the theory the parameters should be constant across dates. The one-state-variable model of Cox, Ingersoll and Ross (1985), nested in Longstaff and Schwartz (1992), has been subject to several tests, e.g. by Brown and Dybvig (1986), Gibbons and Ramaswamy (1993) and Brown and Schaefer (1994).

are outlined in Sections III and IV. Section V presents the data and discusses the method of comparison and the details of the estimation. Section VI reports the results and Section VII concludes.

II. Estimation of Spot and Forward Rates

First we restate definitions and the simple algebra of yields to maturity, spot rates and forward rates; see e.g. Shiller (1990). Consider a coupon bond with a principal of 100 Swedish kronor, an annual coupon c (measured as a proportion of the principal), a time to maturity m (measured in years), and a price p in Swedish kronor (net of accrued interest rate). The annually compounded yield to maturity is the annually compounded internal rate of return that makes the present value of the coupon payments and principal equal to the price of the bond. Formally, let t_k, $k = 1, 2, \ldots, K$, denote the times for the coupon payment, where K is the number of coupon payments. In the special case where m is an integer, we simply have $t_k = k$ and $K = m$. In the general case we have $t_k = m - [m] + k - 1$ and $K = [m] + 1$, where $[m]$ denotes the largest integer that is strictly smaller than m. The yield to the maturity and the price of the bond are then related according to

$$p \equiv \sum_{k=1}^{K} \frac{100c}{(1+y)^{t_k}} + \frac{100}{(1+y)^{t_K}}, \tag{1}$$

where the last term on the r.h.s. is the present value of the principal of the bond.

Let $d(m)$ denote the price of a zero-coupon bond with a principal of 1 krona and a time to maturity of m years. The spot rate is the yield to maturity on a zero-coupon bond. Expressing spot and forward rates as continuously compounded rates, the spot rate $s(m)$ and the price of the discount bond $d(m)$ are related according to $d(m) \equiv \exp[-s(m)m]$ and $s(m) \equiv -[\ln d(m)]/m$. Let $f(m, M)$ denote the (implied) forward rate with settlement in m years and maturity in $M > m$ years. It fulfills $f(m, M) \equiv -[\ln d(M) - \ln d(m)]/(M-m) \equiv [s(M)M - s(m)m]/(M-m)$. The instantaneous forward rate $f(m)$ with settlement (and maturity) in m years is defined as

$$f(m) \equiv \lim_{M \to m} f(m, M) \equiv -\partial \ln d(m)/\partial m \equiv s(m) + m\, \partial s(m)/\partial m.$$

It follows that the spot rate for a given maturity is the average of the instantaneous forward rates with settlement between zero and the spot rate's maturity. Furthermore, it can be shown that the limits of the spot rate and instantaneous forward rate when the maturity approaches zero (denoted $s(0)$ and $f(0)$, respectively) are equal, that is, $s(0) = f(0)$.

The problem of estimating spot and forward rates can then be stated as follows. For a given trade date, let there be n coupon bonds, where bond $j = 1, \ldots, n$, is represented by the triple (c_j, m_j, p_j) of the coupon c_j, the time to maturity m_j and the price p_j, calculated from (1). Let the discount function be modeled by a particular functional form $d(m; b)$, where b is a vector of parameters. The model price of each bond (net of accrued interest), $P_j(b)$, is the present value of the bond when the coupon payments and the principal value are priced with the discount function,

$$P_j(b) \equiv \sum_{k=1}^{K_j} 100 \cdot c_j d(t_{jk}; b) + 100 \cdot d(t_{jK_j}; b), \qquad j = 1, \ldots, n, \tag{2}$$

where t_{jk}, $k = 1, \ldots, K_j$ denotes the times of the coupon payments on bond j. The observed price is assumed to differ from the model price by an error term with zero expectations

$$p_j = P_j(b) + \varepsilon_j, \qquad \mathrm{E}[\varepsilon_j] = 0. \tag{3}$$

The error term can be motivated by institutional features. The yield spread in the data base from Sveriges Riksbank that we use is constructed by taking the best bid and the best ask yield at closing time, hence constructing a minimum bid-ask spread. This procedure may incorporate some mispricing. For example, the yields collected do not necessarily reflect trades at the same time. In addition, yield volatility is usually especially high at closing time, perhaps due to temporary imbalances in supply and demand.[2] The model prices are then fitted to the actual prices using nonlinear least squares or with maximum likelihood (assuming that the error terms are normally distributed).

This approach has been used by several authors, with different functional forms for the discount function.[3] McCulloch (1971, 1975) used a quadratic and cubic spline, respectively. The latter has become a standard method. McCulloch's formulation has the advantage that the estimation can be formulated as a simple linear regression. However, a disadvantage of the cubic spline is that estimates of forward rates may be unstable. Especially for the longest maturities in the sample they may be either very

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[2] Below we shall see that the mean absolute price error is not larger than the average bid-ask spread in the market.

[3] Svensson (1993) discusses simpler, but also inferior approximations of discount functions. Tanggaard (1992) suggests a nonparametric kernel smoothing procedure to estimate the discount function.

large or very small, sometimes even negative.[4] This is a drawback, especially if the focus is on the forward rate estimates, as in monetary policy analysis. The same problem arises for exponential splines and polynomials, which implies that spot and forward rates for long maturities reach large positive or negative values.[5] In addition, the estimates with the cubic spline technique depend on the location of the knot points between different segments.

From an economics point of view it seems reasonable, though, that spot and forward rates for long maturities should be positive and approximately constant. The property that spot and forward rates approach a constant for long maturities is shared by several recently suggested functional forms. Nelson and Siegel's (1987) simple functional form has this property, as have several functional forms that are derived from equilibrium models, for instance the complex two-state-variable model of Longstaff and Schwartz (1992). Below, we give details of the functional forms of Nelson and Siegel, and Longstaff and Schwartz.[6]

III. Nelson & Siegel

Nelson and Siegel (1987) assume that the instantaneous forward rate is the solution to a second-order differential equation with two equal roots. Hence it can be written as

$$f(m; b) \equiv \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau}\right) + \beta_2 \frac{m}{\tau} \exp\left(-\frac{m}{\tau}\right), \qquad (4)$$

where $b = (\beta_0, \beta_1, \beta_2, \tau)$ is the vector of parameters. The spot rate can in turn be derived by integrating the forward rate. It is given by

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[4] See Shea (1984), Langetieg and Smoot (1989), as well as the graphs in McCulloch (1990) or the Gauss viewing program that comes with McCulloch and Kwon (1993) for unstable spot and forward rates. Fisher, Nychka and Zervos (1994) provide an excellent recent extension of McCulloch's method, including several attempts at handling the inherent difficulties with the cubic spline method.

[5] See Vasicek and Fong (1982) and Shea (1985) for references to the use of exponential splines, and Chambers, Carleton and Waldman (1984) for the use of a polynomial for spot and forward rates. Schaefer (1981) used Bernstein polynomials which avoids the problem of negative forward rates. Carleton and Cooper (1976) estimated zero-coupon prices without any restriction on continuity, which also implied large fluctuations in spot and forward rates.

[6] Majnoni (1993) compares the functional forms of Cox, Ingersoll and Ross (1985) and Longstaff and Schwartz (1992) on Italian data. The functional form of Nelson and Siegel (1987) is used, for instance, by Cecchetti (1988) and Green and Oedegaard (1993).

$$s(m;b) \equiv \beta_0 + (\beta_1+\beta_2)\frac{1-\exp(-m/\tau)}{m/\tau} - \beta_2 \exp\left(-\frac{m}{\tau}\right). \tag{5}$$

The spot and forward rates have convenient properties. The limits of the spot and forward rates when maturity approaches zero and infinity, respectively, are $f(0;b) = s(0;b) = \beta_0 + \beta_1$ and $f(\infty;b) = s(\infty;b) = \beta_0$. Thus, the spot and forward rates approach a constant for long maturities and settlements.

Furthermore, suppose there exists a stationary point for the forward rate. That is, suppose there exists $\hat{m} \geq 0$ such that $\partial f(\hat{m};b)/\partial m = 0$, and let $\hat{f} = f(\hat{m};b)$. Then

$$\hat{f} = \beta_0 + \beta_2 \exp\left(-1 + \frac{\beta_1}{\beta_2}\right) \tag{6}$$

and

$$\frac{\hat{m}}{\tau} = 1 - \frac{\beta_1}{\beta_2}. \tag{7}$$

Thus, given β_0 and β_1, β_2 is determined by (6) if there exists a maximum or a minimum $\hat{f}$. Furthermore, since the second derivative of $f(m;b)$ at $\hat{m}$ fulfills

$$\frac{\partial^2 f(\hat{m};b)}{\partial m^2} = -\beta_2 \frac{1}{\tau^2} \exp\left(-\frac{\hat{m}}{\tau}\right), \tag{8}$$

it follows that the sign of β_2 determines whether there is a maximum or a minimum, a negative (positive) β_2 corresponds to a minimum (maximum). Finally, given β_1 and β_2, τ is determined by $\hat{m}$ according to (7). That is, β_0, β_1, β_2, and τ are determined recursively in order by $f(\infty)$, $f(0)$, $\hat{f}$, and $\hat{m}$. The parameters are therefore rather intuitive, and it is easy to find suitable starting values for the optimization procedure.

The NS discount function is then given by $d(m;b) \equiv \exp[-s(m;b)m]$, where $s(m;b)$ is given by (5). The NS forward rate is a very simple functional form; it can have at most one stationary point.

IV. Longstaff & Schwartz

Longstaff and Schwartz (1992) specify a model where there are two state variables, the instantaneous spot rate r and the spot rate's instantaneous rate of variance V. The two state variables are assumed to follow mean-reverting stochastic processes. Their model can be seen as a version with two state variables of Cox, Ingersoll and Ross's (1985) model with one state variable. Longstaff and Schwartz then derive an equilibrium discount function as a solution to a partial differential equation. The solution is a very complex functional form,

$$F(m; r, V) \equiv A(m)^{2\gamma} B(m)^{2\eta} \exp\,[\kappa m + C(m)r + D(m)V], \tag{9}$$

where

$$A(m) \equiv \frac{2\varphi}{(\delta+\varphi)(\exp(\varphi m)-1)+2\varphi}, \tag{10}$$

$$B(m) \equiv \frac{2\psi}{(\nu+\psi)(\exp(\psi m)-1)+2\psi}, \tag{11}$$

$$C(m) \equiv \frac{\alpha\varphi(\exp(\psi m)-1)B(m)-\beta\psi(\exp(\varphi m)-1)A(m)}{\varphi\psi(\beta-\alpha)}, \tag{12}$$

$$D(m) \equiv \frac{\psi(\exp(\varphi m)-1)A(m)-\varphi(\exp(\psi m)-1)B(m)}{\varphi\psi(\beta-\alpha)}, \tag{13}$$

and where $\varphi = \sqrt{2\alpha+\delta^2}$, $\psi = \sqrt{2\beta+\nu^2}$ and $\kappa = \gamma(\delta+\varphi)+\eta(\nu+\psi)$. The parameters α, β, γ, η, δ and ν are functions of the parameters of the stochastic processes for the state variables r and V and the investors' risk aversion. The parameters must be nonnegative, except ν which may be of either sign. The state variables and the parameters α and β must fulfill the restrictions

$$\alpha < \frac{V}{r} < \beta \text{ or } \beta < \frac{V}{r} < \alpha. \tag{14}$$

Considered as a functional form for the discount function as a function of the time to maturity, the state variables are also regarded as parameters and the parameter vector is hence $b = (r, V, \alpha, \beta, \gamma, \eta, \delta, \nu)$. The discount function is thus given by $d(m; b) \equiv F(m; r, V, \alpha, \beta, \gamma, \eta, \delta, \nu)$.

The spot rate can be found from the relation $s(m; b) = -[\ln d(m; b)]/m$ and the instantaneous forward rate can be derived according to the relation $f(m; b) = -\partial \ln d(m; b)/\partial m$. They have the following properties $f(0; b) = s(0; b) = r$ and $f(\infty; b) = s(\infty; b) = \gamma(\varphi-\delta)+\eta(\psi-\nu)$. That is, the spot and forward rates approach a constant for long maturities and settlements.

The LS functional form has some theoretical support, since it is derived from an equilibrium model. The form is very flexible and the forward rate can have both a maximum and a minimum.

V. Data and Method

The data consist of daily data from November 23, 1992 (two business days after the krona was floated on November 19, 1992) to June 21, 1993 — 142 trade dates altogether. Each trade date has observations of the so-called marginal lending rate (the rate at which Sveriges Riksbank lends overnight reserves to banks), the yields on the 11–12 outstanding Swedish Treasury bills, and the yields on the 6–7 outstanding Government benchmark bonds. The data were collected from the data base at Sveriges Riksbank.[7]

The parameters of the discount function were estimated for each trade date separately. That is, the parameters were allowed to change between trade dates. Three different cases, denoted NS1, NS2 and LS were estimated.

Case NS1 refers to estimation of the Nelson and Siegel discount function when the spot and forward rates for zero maturity/settlement, $s(0)$ and $f(0)$, are restricted to be equal to the marginal lending rate. This restriction was imposed to make the estimates comparable to the estimates with case LS, where this restriction was also imposed. In practice, NS1 amounts to imposing the restriction that the sum of the parameters β_0 and β_1 equals the marginal lending rate. Case NS2 denotes estimation of the Nelson and Siegel discount function without the above restriction, in which case $s(0)$ and $f(0)$ may deviate from the marginal lending rate.

Case LS refers to estimation of the Longstaff and Schwartz discount function. The discount function then has 8 parameters, the two state variables r and V and the 6 parameters α, β, γ, η, δ and ν. Since the two state variables can be interpreted as the overnight rate and the volatility of the overnight rate, respectively, they can in principle be estimated separately, or estimated jointly with the other parameters. Because of convergence difficulties and indications that the functional form is overparameterized, we preferred to estimate the state variable separately, in order to reduce

[7] The yields are closing rates, constructed from a minimum bid-ask spread. On average the yield spreads for lower maturities are about 5 basis points (hundredths of a percent), while for longer maturities they are about 3 basis points. The Treasury bills are pure discount bonds, whereas the benchmark bonds pay coupons annually. Arbitrage in the interbank market makes the interbank overnight interest rate close to the marginal lending rate.

the number of parameters. We simply imposed the restriction that r equals the overnight rate. As for the volatility V, we tried to estimate it as a GARCH process for the volatility of 1-week, 1-month and 3-month Treasury bill rates.[8] We could not reject the hypothesis that the volatility was constant, however. Furthermore, some experimenting with different inputs for the volatility for some trading dates revealed that the estimates were rather insensitive. Therefore, throughout the sample we restricted V to equal 0.0010 per year^3, which is about the average volatility for these rates.[9] Thus we ended up estimating the remaining six parameters α, β, γ, η, δ and ν, taking into account the nonnegativity constraints on all except ν, and taking into account the restriction (14).

The restriction that the estimated spot and forward rates should go through the marginal lending or the overnight rate can be motivated by the fact that fitting model prices to observed bond and bill prices gives a low weight to the fit of short-term yields, since the prices are insensitive to the yields for short maturities.[10] The restriction can then be seen as a way of compensating for the low weight in the fit given to the short yields. In our case the low weight is compensated for to some extent because we have relatively many T-bills and relatively few bonds in the Swedish sample. Another possibility would be to experiment by imposing weights on the errors between model and observed prices that decrease at different rates with the time to maturity.[11] Finally, yield errors could be minimized instead of price errors in (3).

NS1 and NS2 were estimated with maximum likelihood, including heteroskedasticity-consistent estimates of the covariance matrix of the parameters; see White (1982). For LS we usually encountered difficulties in the computation of the covariance matrix for the parameters, probably due to flatness of the objective function near the optimum. The reported

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[8] Since the marginal lending rate is held constant between the instances at which it is changed by the central bank, we thought that the volatility in the LS model better corresponds to the volatility of short rates with maturities between 1 week and 3 months.

[9] Since interest rates have the dimension per year, the variance of an interest rate has the dimension per year^2, and V, the instantaneous rate of variance of an interest rate, has the dimension per year^3. The variance of the 1-week, 1-month and 3-month interest rates were between 8 and 12 basis points per year^2.

[10] Using the marginal lending rate or more market determined interest rates, such as the short-term interest rates in the interbank market, actually makes no difference. The marginal lending rate and the overnight interest rate in the interbank market are closely related. Furthermore, the overnight interest rate and interest rates with one to two weeks to maturity are at about the same level, though their volatility differs. There are, for instance, fewer than 10 observations (for each series) that deviate more than 30 basis points from the marginal lending rate during the sample period.

[11] See e.g. Coleman, Fisher and Ibbotson (1992), Langetieg and Smoot (1989) and Majnoni (1993) for examples of different weighting of the error terms.

LS estimates were then estimated with nonlinear least squares, which here gives the same point estimates as maximum likelihood (due to the normality assumption).

As mentioned, the three different cases were then evaluated in terms of measures of fit and convergence properties. Note that even if the LS form has a higher degree of freedom (more parameters) than NS1 and NS2, it is not necessarily so that LS will yield better measures of fit since the models are not nested.

VI. Results

We start by discussing two examples, the estimates for the trade dates November 23, 1992, and April 16, 1993. Figures 1a–c show actual yields to maturity and estimated spot and forward rates for April 16, 1993, estimated for the three cases NS1, NS2 and LS. The squares are observed yields to maturity (percent per year, annually compounded) for the marginal lending rate, 12 Treasury bills and 7 Government bonds. The dashed curves show the estimated spot rates, the solid curves the estimated instantaneous forward rates, and the thin horizontal dashed lines show the infinite-maturity spot and forward rates (that is, $s(\infty)$ and $f(\infty)$). Figures 2a–c show observed T-bill and bond prices (squares), estimated prices (dots), and 100 plus coupon rates in percent (pluses), for the same date.

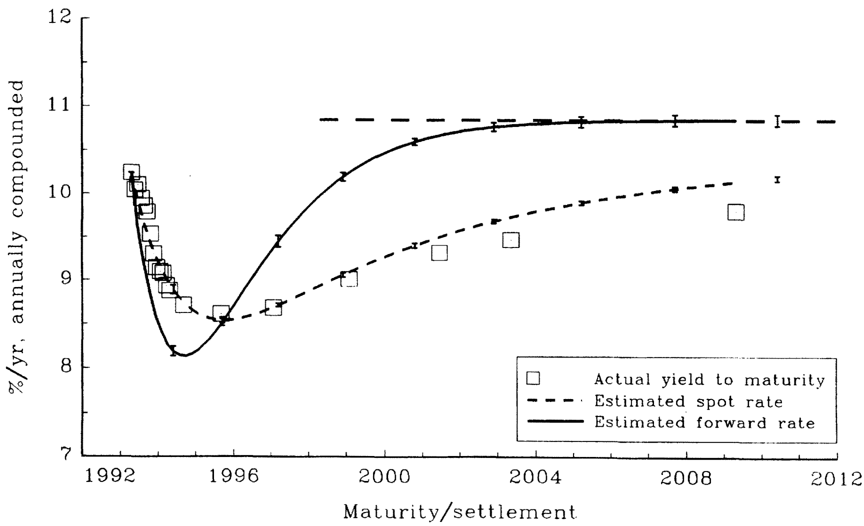

Fig. 1a. NS1 (Nelson and Siegel, w/ restriction), April 16, 1993 — 95 percent confidence interval.

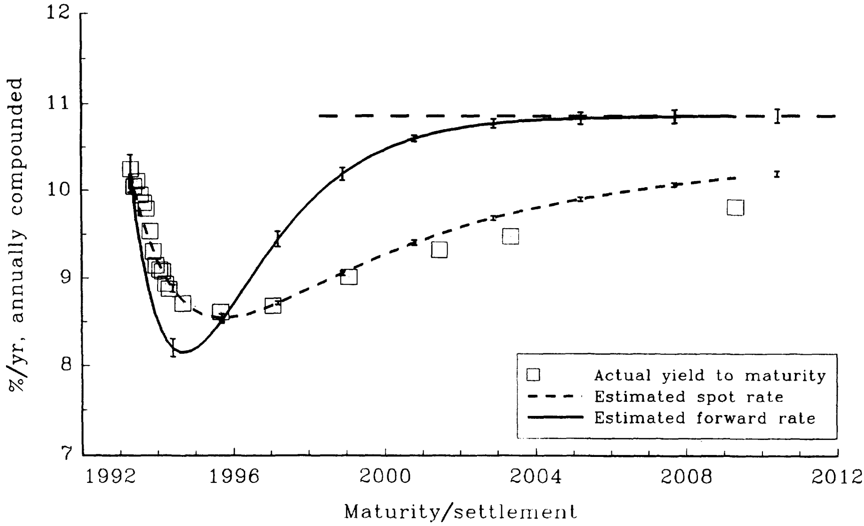

Fig. 1b. NS2 (Nelson and Siegel, w/o restriction), April 16, 1993 — 95 percent confidence interval.

The error bars in the NS figures are 95 percent confidence intervals, computed using the delta method.

Table 1 reports the parameter estimates and measures of the fit for the two dates. Standard errors are included for NS1 and NS2. We see that the

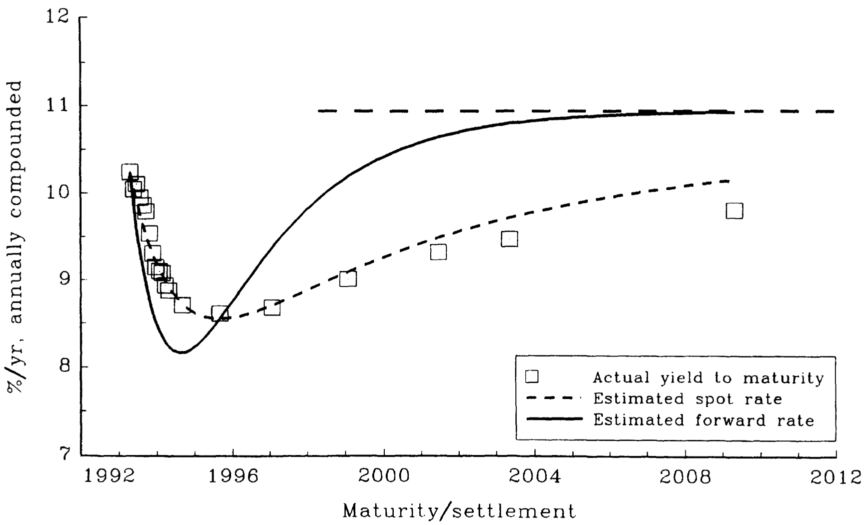

Fig. 1c. LS (Longstaff and Schwartz, w/ restriction), April 16, 1993.

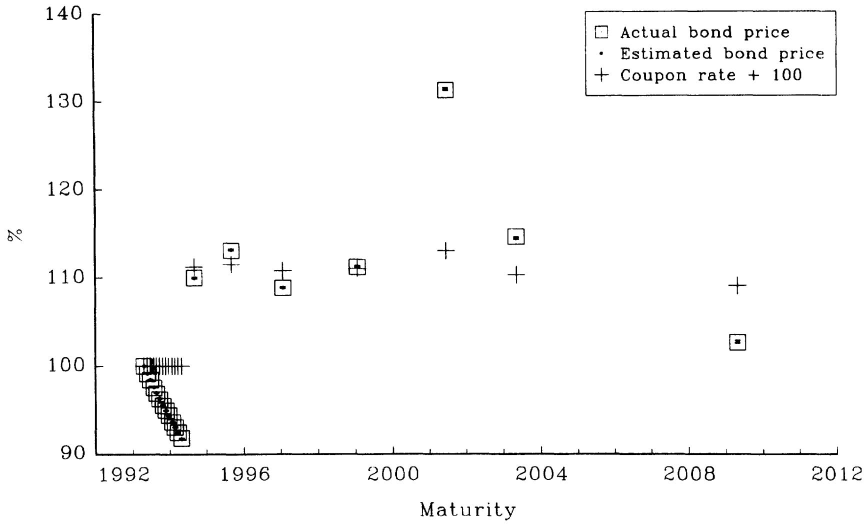

Fig. 2a. NS1, actual and estimated bond prices, April 16, 1993.

NS parameters are estimated fairly precisely. The resulting confidence intervals for the spot and forward rates are also rather narrow, as seen in the figures. It is apparent from the figures that the NS and LS estimates are very similar. Both spot and forward rates are close. For spot rates, the

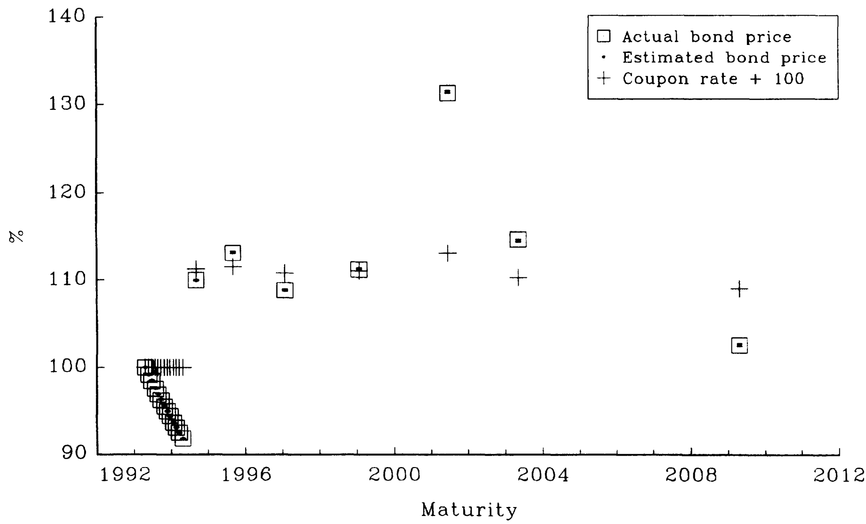

Fig. 2b. NS2, actual and estimated bond prices, April 16, 1993.

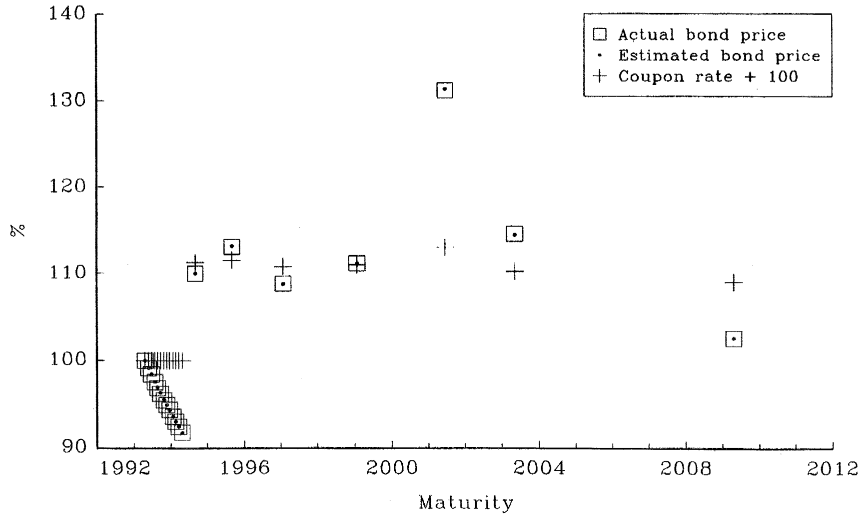

Fig. 2c. LS, actual and estimated bond prices, April 16, 1993.

largest differences occur at the shortest end, and depend on whether or not the NS estimation is restricted to coincide with the marginal borrowing rate. For forward rates, the differences are evenly distributed across different settlements and the LS estimates are within the confidence interval of the NS estimates. In Figures 2a–c the fit of the estimated prices is very good in all cases and differences between the estimates are hardly visible. The root mean squared price errors (RMSPEs) in the three cases vary between 6 and 9 basis points (hundredths of a percent of the principal of the bond); the mean absolute price errors (MAPEs) vary between 6 and 7 basis points. For the yields, the RMSYEs vary between 12 and 23 basis points per year; the MAYEs vary between 9 and 15 basis points per year. Recall that the NS forms are not nested within the LS and the reported measures of fit are all adjusted for the degree of freedom (3, 4 and 6 for the NS1, NS2 and LS, respectively).

The fit of LS is marginally better. In terms of RMSPE the fit of NS2 (without the restriction) is naturally better than that of NS1 (with the restriction). The difference is rather small though. The yield errors are similar, except for NS2 on November 23, due to the errors for the shortest maturities. Whether or not the restriction is imposed in the NS estimation indeed sometimes leads to large differences (sometimes 100 basis points per year) in estimated spot and forward rates for the shortest maturities. This is due to the fact that minimizing price errors gives little weight to short yields (since prices are insensitive in short yields). The differences between the estimates on November 23 are larger than is typical, whereas the differences between the estimates on April 16 are typical for the rest of the sample.

Finally, we see in Table 1 that the convergence properties are rather different. NS1 and NS2 converge with few iterations in a short time. LS needs many more iterations, and even more time, since each iteration takes longer.

Next we discuss summary results for the three cases based on the whole sample. With regard to the fit, we see in Table 2 that the (sample) means of the MAPE and RMSPE are highest for NS1, 9 and 11 basis points, respectively, with (sample) standard deviations of 3 basis points and

Table 1. *Estimation results, November 23, 1992 and April 16, 1993*

	NS1		NS2		LS	
	(1) Nov 23	(2) Apr 16	(3) Nov 23	(4) Apr 16	(5) Nov 23	(6) April 16
Parameters						
β_0 (%/yr)	10.24	10.29	10.32	10.30		
	(0.05)	(0.03)	(0.05)	(0.04)		
β_1 (%/yr)	2.26	−0.54	1.58	−0.59		
			(0.24)	(0.13)		
β_2 (%/yr)	−5.76	−6.16	−5.11	−6.08		
	(0.24)	(0.06)	(0.31)	(0.19)		
τ (yrs)	0.87	1.53	1.04	1.55		
	(0.04)	(0.04)	(0.10)	(0.07)		
α					0.0061	0.0000
β					0.0102	0.0260
γ					8.4211	0.0042
δ					2.5153	1.0899
η					2.6390	1.4348
v					0.3001	0.3234
r (%/yr)					12.50	9.75
V (b.p./yr^3)					10.00	10.00
Measures of fit						
RMSPE (%)	0.0896	0.0730	0.0736	0.0748	0.0643	0.0741
MAPE (%)	0.0688	0.0607	0.0614	0.0644	0.0632	0.0678
RMSYE (%/yr)	0.1297	0.0964	0.2346	0.1121	0.1166	0.1160
MAYE (%/yr)	0.0898	0.0732	0.1463	0.0854	0.0921	0.0878
Convergence						
Iterations	10	18	14	15	501	377
Time (minutes)	0.05	0.14	0.10	0.14	10.43	10.32

Note: NS1, NS2 and LS refer to Nelson and Siegel with restriction, without restriction and Longstaff and Schwartz with restriction, respectively. Heteroskedasticity-consistent standard errors for the parameters in columns (1)–(4) are given in parentheses. RMSPE and MAPE denote the root mean square price error and mean absolute price error, respectively, in percent of the principal. RMSYE and MAYE are the analogs for yield errors, in percentage points per year. Iterations and Time denote the number of iterations and the time to convergence.

Table 2. *Summary of fit*

	Mean	Std dev	Min	Max	Obs	Miss
2a. NS1 (Nelson & Siegel with restriction)						
RMSPE	0.1097	0.0302	0.0443	0.2074	142	1
MAPE	0.0897	0.0302	0.0374	0.2060	142	1
RMSYE	0.1547	0.0792	0.0363	0.3954	142	1
MAYE	0.1148	0.0568	0.0298	0.3091	142	1
2b. NS2 (Nelson & Siegel without restriction)						
RMSPE	0.0841	0.0284	0.0228	0.1782	142	1
MAPE	0.0607	0.0151	0.0177	0.1088	142	1
RMSYE	0.1751	0.1148	0.0329	0.4536	142	1
MAYE	0.1024	0.0554	0.0299	0.2609	142	1
2c. LC (Longstaff & Schwartz with restriction)						
RMSPE	0.0884	0.0371	0.0242	0.1893	142	8
MAPE	0.0731	0.0276	0.0227	0.1808	142	8
RMSYE	0.1386	0.0725	0.0314	0.3711	142	8
MAYE	0.0983	0.0470	0.0325	0.3005	142	8

Note: RMSPE and MAPE (in percent of the principal) denote the root mean square price error and mean absolute price error, respectively. RMSYE and MAYE (in percentage points per year) are the analogs for yield errors. Obs refers to the total number of trade dates, and Miss refers to the number of trade dates for which convergence failed.

(sample) maxima of about 21 points. The means of the MAPE and RMSPE are lower for NS2 and LS, about 6 and 9 basis points, respectively, with standard deviations between 1 and 3 basis points, and maxima between 11 and 19 basis points. The means of the MAYEs vary between 10 and 11 basis points per year, with standard deviations between 5 and 6 basis points per year. Altogether LS appears to have at most a marginally better fit than NS1 but no better fit than NS2. The maximum MAPE is about 21 basis points, and the maximum MAYE does not exceed 40 basis points per year, which indicate a precision more than sufficient for monetary policy analysis.

With regard to the absolute deviations in estimated spot and forward rates between NS1 and LS, reported in Table 3, we see that the estimated spot rates are very similar. The means for the differences in the spot rates with 1–10 years to maturity are between 2 and 5 basis points per year. The largest differences occur at the shorter maturities. The maxima are about 20 basis points per year for maturities longer than one year. For the forward rates, the average differences are slightly larger, 7–11 basis points per year, and they seem to be uniformly distributed over the maturity. The standard deviations are larger and the greater variability can also be seen in the minimum and maximum columns. However, the maxima do not exceed 40 basis points per year for maturities less than 10 years.

Table 3. *Summary of absolute differences in spot and forward rates*

	Mean	Std dev	Min	Max	Obs	Miss
Maturity						
3a. Spot rates (NS1-LS)						
0.5	0.0973	0.0835	0.0008	0.3793	142	8
1	0.0485	0.0460	0.0003	0.2028	142	8
3	0.0414	0.0326	0.0004	0.1468	142	8
5	0.0192	0.0169	0.0003	0.0681	142	8
7	0.0221	0.0192	0.0000	0.0730	142	8
10	0.0187	0.0145	0.0002	0.0569	142	8
Settlement						
3b. Forward rates (NS1-LS)						
0.5	0.0763	0.0673	0.0024	0.3080	142	8
1	0.0812	0.0969	0.0001	0.3901	142	8
3	0.0702	0.0642	0.0009	0.3401	142	8
5	0.1014	0.0787	0.0002	0.3133	142	8
7	0.0649	0.0521	0.0002	0.2527	142	8
10	0.1135	0.1501	0.0008	0.6452	142	8

Note: Summary statistics for absolute differences (in percentage points per year) in estimated spot and forward rates. NS1 and LS refer to Nelson and Siegel with restriction and Longstaff and Schwartz with restriction, respectively. Maturity and settlement are measured in years. Obs refers to the number of trade dates, and Miss refers to the number of trade dates for which convergence failed.

According to the measures of fit, LS appears to have a marginally better fit than NS. However, the price and yield errors should be related to the normal price and yield spread in the market. Swedish bonds are quoted in yields to maturity. A typical yield spread is about 3 basis points, which results in bond price spreads of between 5 and 20 basis points, depending on the coupon rate and the time to maturity. Hence, the yield and price errors reported above are of the same order of magnitude as the yield and price spreads, and sometimes within the yield and price spreads. Therefore we conclude that LS has at most a marginally better fit, but that the small improvement in the fit has no practical implication since it is within the normal yield and price spreads.

With regard to the parameter estimates, we see in Table 4 that for NS1 and NS2 the means of β_0, β_1 and β_2 are about 10, -0.4 and -6 percent per year (the dimension of the betas). (The mean of $f(\infty)$ exceeds the mean of β_0 for NS1 and NS2 simply because the first is annually and the second is continuously compounded.) The corresponding standard deviations are between 0.4 and 1.3 percent per year. The mean of τ is about 1.4 years with standard deviation of about 0.3 year.

For the parameters of LS, the standard deviations are much larger relative to the means. The estimates of the parameters are in that sense rather unstable. Moreover, similar estimated spot and forward curves have rather different parameters. This indicates that the correlation between the estimates may be high, and that the model may be overparameterized. It has not been possible to verify this by estimating the covariance matrix of the estimates for LS. However, the circumstance that the covariance matrix is difficult to compute is in itself an indication of overdetermination. There is, of course, no presumption that the parameters should be constant during the sample and the means reported in Table 4 only indicate the average magnitude of the parameters.

The (sample) mean of the asymptote $f(\infty)$ is about 100 basis points per year lower for LS than for NS1 and NS2. The (sample) standard deviation is larger for LS than for NS1 and NS2: 43 basis points per year for NS1 and NS2, and almost 8 times larger for LS. This result arises since in a few cases the LS estimation results in a very low estimate of $f(\infty)$. Consistent with this, the medians of the estimates of $f(\infty)$ are similar, 10.65, 10.66 and 10.60 percent per year for NS1, NS2 and LS, respectively. Nevertheless, the parameter estimates of NS, including the asymptotic spot and forward rate $f(\infty)$, are much more stable than those of LS. This, together with their clear interpretation, is certainly an advantage for NS.

Table 4. *Summary of parameter estimates and convergence*

	Mean	Std dev	Min	Max	Obs	Miss
4a. NS1 (Nelson & Siegel with restriction)						
β_0 (%/yr)	10.11	0.39	9.43	10.79	142	1
β_2 (%/yr)	−5.92	0.91	−7.23	−3.02	142	1
τ (yrs)	1.31	0.30	0.56	1.94	142	1
$f(\infty)$ (%/yr)	10.64	0.43	9.89	11.40	142	1
Iterations	16.05	4.51	8.00	34.00	142	1
4b. NS2 (Nelson & Siegel without restriction)						
β_0 (%/yr)	10.16	0.39	9.46	11.22	142	1
β_1 (%/yr)	−0.42	1.00	−1.61	3.59	142	1
β_2 (%/yr)	−5.47	1.26	−8.13	−3.19	142	1
τ (yrs)	1.46	0.26	0.48	2.01	142	1
$f(\infty)$ (%/yr)	10.70	0.43	9.92	11.87	142	1
Iterations	27.70	22.65	13.00	205.00	142	1
4c. LS (Longstaff & Schwartz with restriction)						
α	0.01	0.02	0.00	0.17	142	8
β	0.05	0.05	0.00	0.22	142	8
γ	2.81	5.64	0.00	27.10	142	8
δ	1.43	1.21	0.00	5.56	142	8
η	1.67	3.07	0.00	13.76	142	8
ν	0.53	1.04	−0.94	3.83	142	8
$f(\infty)$ (%/yr)	9.60	3.20	0.00	14.91	142	8
Iterations	538.34	436.60	32.00	2458.00	142	8

Note: Obs refers to the total number of trade dates, and Miss refers to the number of trade dates for which convergence failed.

With regard to the convergence properties, the convergence for both NS1 and NS2 is relatively insensitive to starting values and occurs with relatively few iterations. Each iteration is also quick, so in general convergence is fast and easy. NS1 converges even more easily than NS2, on average 16 against 28 iterations, and a maximum 38 iterations against 250. The average time for each iteration is about 0.5 second on a 486 machine with 50 MHz clock frequency. From this point of view, using NS is fast and easy. In contrast, convergence in LS is extremely sensitive to starting values, and frequently requires very many iterations. Many iterations are very slow and local minima abound. The average number of iterations is more than 500, sometimes 2000 iterations are required. The average time for each iteration is about 1.5 seconds, but each iteration frequently takes up to 8 seconds. Convergence failed in 8 cases.

VII. Conclusions

We have estimated the Swedish term structure with two functional forms, the simple form of Nelson and Siegel (1987) and the complex form of Longstaff and Schwartz (1992). The functional forms have been compared with regard to their performance in estimating spot and forward interest rates to be used in monetary policy analysis, for instance as monetary policy indicators.

The result of our comparison is that LS has a marginally better fit, but that NS is superior in terms of convergence properties, confidence interval computation, parameter stability and parameter interpretation. Furthermore, the NS fit seems well above what is needed for monetary policy analysis. On balance, our comparison thus favors NS.

The comparison is made on the Swedish term structure between November 1992 and June 1993. It appears that this term structure was not sufficiently complicated to warrant the flexibility of LS. This does not, of course, exclude the possibility that the term structure on other occasions and for other countries could be too complicated for NS and therefore warrant LS or other more flexible forms. Since the NS functional form only allows for a single interior maximum or minimum, one major determinant of whether NS gives a good fit or not should be whether the term structure has more than one interior maximum or minimum. It remains an open question how often that occurs.[12] In any case, a simple operational way to estimate the term structure is to start with a simple form like NS and then

[12] It is of course possible to compare estimation with NS and LS in a Monte-Carlo study. However, the result of the comparison will then be heavily influenced by what functional form is used to generate the data.

judge whether the fit is sufficiently good. If not, a more complex form should be tried to see if the fit improves.[13]

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[13] A more flexible variant of NS, that allows two rather than one hump or trough but is still very easy to use, has later been presented in Svensson (1994a, 1995).

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