Estimating Forward Interest Rates with the Extended Nelson & Siegel Method

LARS E.O. SVENSSON

Several central banks use implied forward interest rates as one of their monetary policy indicators. The paper outlines a convenient and for monetary policy purposes sufficiently precise method to estimate implied forward rates from Treasury bill and coupon bond data. The method uses an extended and more flexible variant of Nelson and Siegel's functional form. Minimization of both price errors and yield errors is discussed.

The purpose of this paper is to demonstrate a convenient and for monetary policy purposes sufficiently precise method to estimate implied forward interest rates. Forward interest rates are interest rates on investment and loans that start at a future date, the settlement date, and last to a date further into the future, the maturity date.

The paper is motivated by the increased need for monetary policy indicators when flexible exchange rates have replaced fixed exchange rates. The collapse of fixed exchange rates in Europe and the widening of ERM bands mean that a well-defined intermediate target for monetary policy has been lost. In this situation, whether a new intermediate target is introduced or not, the role of indicators will be crucial, both for assessing the state of the economy and the stance of monetary policy, and for deciding whether the instrument of monetary policy is on a path appropriate for achieving the monetary policy goal. Forward interest rates should be suitable as one of the several indicators that need to be used.

The use of forward interest rates has long been standard in financial analysis, for instance in pricing new financial instruments and in discovering arbitrage possibilities. Although yield curves are a standard tool in monetary policy analysis in central banks and elsewhere, the use of forward interest rates for monetary policy purposes appears to be relatively recent. For instance, the Board of Governors of the Federal Reserve System, the Bank of England, and Sveriges Riksbank have recently included forward interest rates among their indicators.[1]

Forward interest rates can, under the assumption of negligible term premia, be interpreted as indicating market expecta-

Lars E.O. Svensson, Professor of International Economics at the Institute for International Economic Studies, Stockholm University, and Advisor to the Riksbank.

The paper is an abbreviated version of Svensson (1994a). I thank Jon Faust for discussions and Mårten Blix for research assistance.

1. See Bank of England (1994, p. 15), Deacon and Derry (1994), Reinhart and Klapper (1992), and Sveriges Riksbank (1993a,b). Cf. also Bank of International Settlement (1993).

tions of future short interest rates. Forward rates contain the same information as the standard yield curve. Indeed, the forward rate curve is related to the yield curve as the marginal cost curve is to the average cost curve. Forward rates present the information in a way more easily interpreted for monetary policy purposes. Whereas the yield curve can be interpreted as expected *averages* of future short rates, the forward rate curve can be interpreted as indicating the expected *time path* of future short rates. Therefore forward rates more easily allow a separation of expectations for the short, medium and long term than the yield curve. This is the main advantage of forward rates.[2]

In the absence of a full set of forward markets, implied forward interest rates need to be estimated from data on existing financial instruments, usually Treasury bills and Government bonds. For financial analysis, the estimation of forward rates is done with a number of different methods, some rather complex in order to achieve sufficient precision. For monetary policy analysis, the need for precision is arguably less and can be traded for increased robustness and simplicity of the estimation method. Here I will present an estimation method that is simple and robust but appears to have a precision well beyond what is needed for monetary policy purposes. The method uses an extension first proposed in Svensson (1993) of the functional form suggested by Nelson and Siegel (1987). The method dominates the methods discussed in Svensson (1994a). It has recently been adopted by the Bank of England (Deacon and Derry (1994)).

2. The analogy with average and marginal cost is exact when interest rates are continuously compounded, and when a zero-coupon yield curve is used. The analogy is approximate for coupon bond yield curves, and for annually compounded interest rates.

In the next section the relations between spot rates (zero-coupon rates), yields to maturity and forward interest rates are defined and discussed. Thereafter comes a presentation of the estimation method, followed by the conclusions. The Appendix discusses some technical aspects of the estimation.

Spot rates, yields to maturity, and forward rates

In the absence of explicit forward markets, implied forward interest rates have to be estimated from interest rates on existing financial instruments. To compute implied forward interest rates from yields to maturity on zero-coupon bonds, spot rates, is easy. To compute implied forward interest rates from yields to maturity on coupon bonds is more complicated. Inconveniently, almost all bonds with time to maturity beyond twelve months are coupon bonds rather than zero-coupon bonds. Yields to maturity on coupon bonds are not identical to yields to maturity on zero-coupon bonds of the same maturity. Since a coupon bond can be seen as a portfolio of zero-coupon bonds of different maturities (each zero-coupon bond corresponding to a particular coupon payment), yields to maturity on coupon bonds are a kind of average of yields to maturity on zero-coupon bonds of maturities from the time of the first coupon payment to the time of the payment of the face value and last coupon. Estimating forward rates from coupon bonds can then be seen as involving two steps: first implied spot rates are estimated from yields to maturity on coupon bonds, and then implied forward rates are computed from implied spot rates.

To understand this more precisely, some algebra is needed. The algebra of spot rates, yields to maturity on coupon bonds, and forward rates is easiest if all rates are continuously compounded.[3] Let $i(t,T)$ (measured in per cent per year) be the continuously compounded spot interest rate for a zero coupon bond traded at time t, the trade date, that matures at time $T>t$, the maturity date. Let $m=T-t$ denote the time to maturity. The term structure of interest rates at a given trade date t is unambiguously represented by a graph of the spot rate $i(t,t+m)$ for different times to maturity m. Let $d(t,T)$, the discount function, denote the price at time t of a zero-coupon bond that pays 1 krona at the maturity date T. It is related to the spot rate by

$$(1) \quad d(t,T) = \exp\left(-\frac{i(t,T)}{100}(T-t)\right),$$

where $\exp(x)$ denotes the exponential function e^x.

Consider now a coupon bond with coupon rate c per cent per year, a bond that pays a face value of 100 kronor at the maturity date but also pays an annual coupon of c kronor each year up to and including the maturity date. Let the time to maturity be m years.[4] The present value at the trade date t of a coupon payment made in year k, $k=1, 2, \ldots, m$, will be $cd(t, t+k)$ and the present value of the face value paid in year m will be $100d(t, t+m)$. It follows that the price of the bond at the trade date, $P(t, t+m)$, will equal

$$(2) \quad P(t,t+m) = \sum_{k=1}^{m} cd(t,t+k) + 100d(t,t+m).$$

For coupon bonds, yields to maturity are often quoted rather than prices. The yield to maturity is the internal rate of return for the coupon bond, that is, the constant interest rate that makes the present value of the coupon payments and the face value equal to the price of the bond. Hence the yield to maturity $y(t, t+m)$ (measured in per cent per year) on the coupon bond fulfills the equation

$$(3) \quad P(t,t+m) = \sum_{k=1}^{m} c\exp\left(-\frac{y(t,t+m)}{100}k\right) + 100\exp\left(-\frac{y(t,t+m)}{100}m\right)$$

Yield curves showing the yield to maturity on coupon bonds for different maturities are frequently used to represent the term structure of interest rates but the picture they provide is imprecise. First, a given yield to maturity can be seen as an average of the spot rates up to the time to maturity. The present values of the coupon payments and face value that are used to price the bond in equation (2) correspond to spot rates in (1) that generally vary with the maturity. In contrast, the present values of the coupon payments and face value that are used to price the bond in equation (3) are computed with a *constant* yield to maturity, which is hence a somewhat complicated average of the spot rates. Second, for a given term structure of spot rates, the yield to maturity for a bond with a given maturity will depend on its coupon rate, the »coupon effect«. Therefore two coupon bonds that mature at the same date

3. The continuously compounded spot rate i and the annually compounded spot rate $\tilde{i}$ (both measured in per cent per year) are related by
$\tilde{i} = 100\left(\exp\left[\frac{i}{100}\right]-1\right)$ and $i = 100\ln\left[1+\frac{\tilde{i}}{100}\right]$. See Shiller (1990) or Fage (1986) for details on the algebra of spot rates, yields to maturity and forward rates.

4. The general case when the time to maturity is not an integer is handled in the Appendix. For semi-annual coupon payments, as in Britain and the United States, the formulas are accordingly modified (see for instance Fage (1986)).

have different yields to maturity if they have different coupon rates. The reason is that, everything else equal, a higher coupon rate implies that the share of early payments increases, which gives more weight to short spot rates in the determination of the yield to maturity.

For these reasons yield curves for coupon bonds should not be used as direct representations of the term structure of interest rates. Instead spot rates should be used.[5] For maturities below twelve months spot rates are directly available in the form of rates on Treasury bills, which are zero-coupon bonds. For longer maturities zero-coupon bonds are usually not available for sufficiently many maturities and in sufficiently large issues to be sufficiently liquid. Therefore, spot rates will have to be estimated from yields on coupon bonds. Also, even if there were quite a few liquid longer-maturity zero-coupon bonds, one might still want to use the additional information in the coupon bond yields.

Implied forward rates are easy to calculate from spot rates, since a forward investment with specific settlement and maturity dates can be reproduced by a sale and a purchase of zero-coupon bonds: a sale of a bond maturing on the forward contract's settlement date and a purchase of bonds of the same market value that matures on the forward contract's maturity date. The implied forward rate is the return on such a readjustment of a bond portfolio.

More precisely, let $f(t,t',T)$ (measured in per cent per year) be the continuously compounded (implied) forward rate on a forward contract concluded at time t, the trade date, for an investment that starts at time $t'>t$, the settlement date, and ends at time $T>t'$, the maturity date. Then the forward rate is related to the spot rates according to

$$(4)\qquad f(t,t',T) = \frac{(T-t)\,i(t,T)-(t'-t)\,i(t,t')}{T-t'}.$$

That is, the forward rate for a one-year investment with settlement in four years ($t'-t=4$ years) and maturity in five years ($T-t=5$ years) (»the one-year forward rate four years from now«) is equal to five times the five-year spot rate minus four times the four-year spot rate.

The instantaneous(-maturity) forward rate is the forward rate for a forward contract with an infinitesimal investment period after the settlement date. It is defined as the limit

$$(5)\qquad f(t,t') = \lim_{T\to t'} f(t,t',T).$$

In practice it can be identified with an overnight forward rate, that is, a forward rate with maturity one day after settlement. The finite-maturity forward rate $f(t,t',T)$, $T>t'$ will be the average of the instantaneous forward rates with settlement between t' and T,

$$(6)\qquad f(t,t',T) = \frac{\int_{\tau=t'}^{T} f(t,\tau)\,d\tau}{T-t'}$$

The instantaneous forward rate can be seen as the marginal increase in the total return from a marginal increase in the length of the investment. Instantaneous forward rates and finite-maturity spot rates are therefore related precisely as marginal and average cost of production, when the time to maturity is identified with quantity produced. The spot rate $i(t,T)$ at time t with maturity at time T is hence identical to the average of the instantaneous forward rates with settlements between the trade date t and the maturity date T,

5. A so-called par yield curve is an alternative way to unambiguously represent the term structure of interest rates.

$$i(t,T) = \frac{\int_{\tau=t}^{T} f(t,\tau)\,d\tau}{T-t} \tag{7}$$

Equivalently, the forward and spot rates fulfil the relation

$$f(t,T) \equiv i(t,T) + (T-t)\frac{\partial i(t,T)}{\partial T}, \tag{8}$$

which is another standard relation between marginal and average cost, with time to maturity $T-t$ corresponding to quantity produced. Hence, when looking at spot and forward rates one often recognizes the shapes of average and marginal curves familiar from microeconomics textbooks.

Estimating forward rates with the extended Nelson and Siegel method

The estimation of spot and forward rates suggested here follows McCulloch (1971, 1975) in fitting for each trade date a discount function (the price of a zero-coupon bond as a function of the time to maturity) to bill and bond price data, but it uses an extension of the functional form of Nelson and Siegel (1987) instead of McCulloch's original cubic spline. The cubic spline has the well-known disadvantage that estimates of forward rates may be rather unstable, especially at the longest maturity (Shea (1984)).[6]

Nelson and Siegel (1987) assumes that the instantaneous forward rate is the solution to a second-order differential equation with two equal roots. Let me simplify the notation by letting $f(m)$ denote the instantaneous forward rate $f(t,t+m)$ with time to settlement m, for a given trade date t. Then Nelson and Siegel's forward rate function can be written

$$\begin{aligned} f(m;b) &= \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) \\ &\quad + \beta_2 \frac{m}{\tau_1}\exp\left(-\frac{m}{\tau_1}\right), \end{aligned} \tag{9}$$

where $b=(\beta_0,\beta_1,\beta_2,\tau_1)$ is a vector of parameters (β_0 and τ_1 must be positive).

The forward rate in (9) consists of three components. The first is a constant, β_0, the second is an exponential term, $\beta_1\exp\left(-\frac{m}{\tau_1}\right)$, monotonically decreasing (or increasing, if β_1 is negative) towards zero as a function of the time to settlement, and the third is a term which generates a hump-shape (or a U-shape, if β_2 is negative) as a function of the time to settlement, $\beta_2\frac{m}{\tau_1}\exp\left(-\frac{m}{\tau_1}\right)$. When the time to settlement approaches infinity, the forward rate approaches the constant β_0, and when the time to settlement approaches zero, the forward rate approaches the constant $\beta_0+\beta_1$.

To increase the flexibility and improve the fit I extend Nelson and Siegel's function by adding a fourth term, a second hump-shape (or U-shape), $\beta_3\frac{m}{\tau_2}\exp\left(-\frac{m}{\tau_2}\right)$,

6. Fisher, Nychka and Zervos (1994) have recently provided several important extensions to McCulloch's cubic spline method. Dahlquist and Svensson (1995) compare the original functional form of Nelson and Siegel (1987) to the much more complex functional form of Longstaff and Schwartz (1992) on Swedish data for the sample period December 1992–June 1993. The Nelson and Siegel functional form is much easier to use than the Longstaff and Schwartz functional form. The additional flexibility of the latter is not needed for that sample period. When additional flexibility is needed, the extended Nelson and Siegel functional form seems preferable since it is much easier to use than the Longstaff and Schwartz functional form.

with two additional parameters, β_3 *and* τ_2 (τ_2 must be positive). The function is then

$$f(m;b) = \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau_1}\right) + \beta_2 \frac{m}{\tau_1} \exp\left(-\frac{m}{\tau_1}\right) + \beta_3 \frac{m}{\tau_2} \exp\left(-\frac{m}{\tau_2}\right), \tag{10}$$

where $b=(\beta_0,\beta_1,\beta_2,\tau_1,\beta_3,\tau_2)$.

The function and its components are illustrated in *Figure 1.* The parameters are $\beta_0 =$ 8.06 per cent per year, $\beta_1 = -0.31$ per cent per year, $\beta_2 = -6.25$ per cent per year, $\tau_1 =$ 1.58 year, $\beta_3 = -1.98$ per cent per year and $\tau_2 = 0.15$ year. (These are the estimates for Sweden for the trade date December 29, 1993.) The thick solid curve is the forward rate curve. The other curves show the four components of the forward rate curve.

The spot rate can be derived by integrating the forward rate according to (7). Let $i(m)$ denote the spot rate $i(t, t+m)$ with time to maturity m, for a given trade date t. It is given by

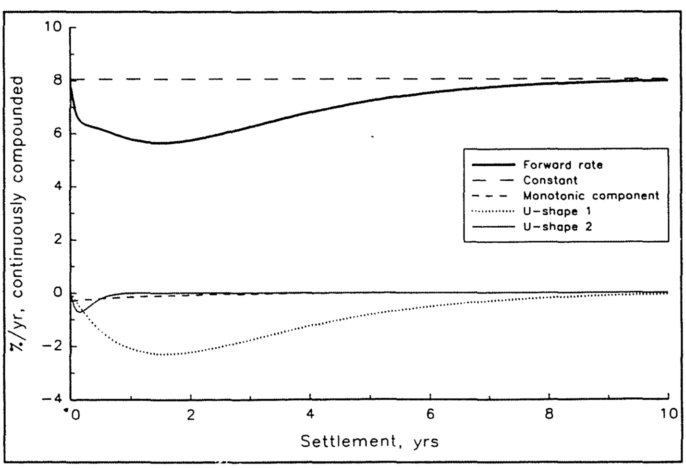
$$i(m;b) = \beta_0 + \beta_1 \frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} + \beta_2 \left(\frac{1-\exp\left(-\frac{m}{\tau_1}\right)}{\frac{m}{\tau_1}} - \exp\left(-\frac{m}{\tau_1}\right) \right) + \beta_3 \left(\frac{1-\exp\left(-\frac{m}{\tau_2}\right)}{\frac{m}{\tau_2}} - \exp\left(-\frac{m}{\tau_2}\right) \right) \tag{11}$$

The discount function is then given by

$$d(m;b) = \exp\left(-\frac{i(m;b)}{100} \mathrm{m}\right) \tag{12}$$

The discount function is estimated for each trade date by minimizing either (the sum of squared) price errors or (the sum of squared) yield errors. Let me first discuss minimizing price errors. Then, for given parameters the discount function is used to compute estimated bond prices according

FIGURE I. *Forward Rate Curve and Components*



to (2). The parameters are chosen so as to minimize the sum of squared errors between the estimated and observed prices of the bonds, where the observed prices of the bonds are calculated from quoted yields to maturity, coupon rates and times to maturity according to (3). This is the standard way since McCulloch (1971, 1975).

Minimizing price errors sometimes results in fairly large yield errors for bonds and bills with short maturities. This is because prices are very insensitive to yields for short maturities.[7] Then it may be better to choose the parameters so as to minimize yield errors. One can also argue that since in monetary policy analysis the focus is on interest rates rather than prices, it makes sense to minimize errors in the yield dimension rather than in the price dimension. When yield errors are minimized, for given parameters the discount function is still used to compute estimated prices of coupon bonds according to (2). Then the estimated yield to maturity for each bond is computed by solving (3). The parameters are chosen so as to minimize the sum of squared yield errors between estimated yields and observed yields.

The estimation is conveniently done with Maximum Likelihood, although Nonlinear Least Squares or the Generalized Method of Moments can also be used. Heteroskedasticity-consistent standard errors for the parameters are estimated and used with the so-called delta method to obtain confidence intervals for spot and forward rates. Details of the estimation method are given in the Appendix.

In some cases minimizing price errors results in quite good fits of yields. In many cases, however, the yield fit for short maturities appears less than satisfactory, and minimizing the yield errors then gives a much better fit, usually with only a minor deterioration of the price fit.

In many cases the original Nelson and Siegel model gives a satisfactory fit. But in some cases when the term structure is more complex the fit is unsatisfactory. Then the extended model improves the fit considerably.

As an example of an estimation result, see *Figure 2*. It shows the estimate for Sweden for the trade date December 29, 1993. The estimation is done with minimized yield errors, for the extended Nelson and Siegel model. The squares show the observed marginal lending rate and observed yields to maturity on Treasury bills and Government benchmark bonds, plotted against the maturity date.[8] All rates are annually compounded. The plus signs show the coupon rates for the bonds (the Treasury bills and the marginal lending rate have zero coupons). The dashed curve shows the estimated spot rate curve, the zero-coupon rates. The error bars show 95 per cent confidence intervals. Dots with error bars show the estimated yields to maturity with 95 per cent confidence intervals (the estimated yields are on the spot rate curve for zero-coupon bonds but are generally not so for coupon bonds, since yields to maturity on coupon bonds gener-

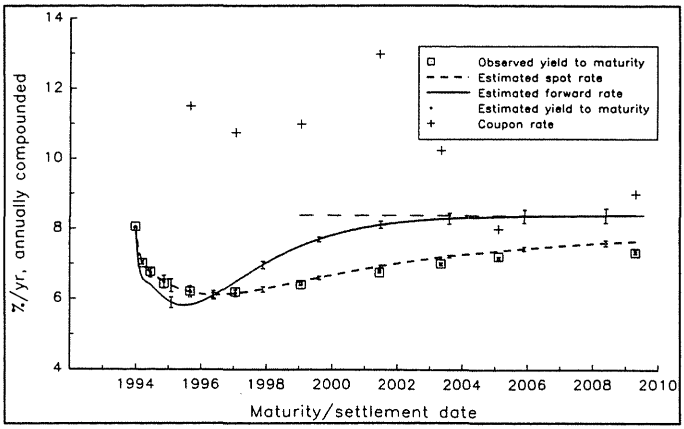
7. Recall that the elasticity of the price with respect to one plus the yield is equal to the duration of a bond (the present-value weighted average maturity of coupon payments and face value).

8. The marginal lending rate was, through May 1994, the overnight rate at which banks could borrow reserves from the Riksbank. It can be seen as the Riksbank's monetary policy instrument. Arbitrage by banks implied that the interbank overnight rate would be close to the marginal lending rate.

From June 1, 1994, the Riksbank uses a system similar to that of Bundesbank, in that the repo rate, bounded by a floor (the deposit rate) and a ceiling (the lending rate), serves as the policy instrument.

FIGURE 2. *Minimized Yield Errors, Extended N&S. Yields to Maturity, Spot and Forward Rates, Sweden, 29 December 1993*

95 % confidence interval



ally differ from yields to maturity on zero-coupon bonds). The fit is good, and the error bars are hardly visible. The solid curve shows the estimated (instantaneous) forward rate, plotted against the settlement date, with error bars showing 95 per cent confidence intervals. The horizontal dashed line is the asymptote for the spot and forward rate (the parameter β_0). The root mean squared yield error for this estimation is 0.03 percentage points per year, the root mean squared price error is 0.16 kronor for a bond with face value 100 kronor.

On this date the spot rate curve has a U-shape similar to the standard shape of average cost curves in microeconomics textbooks. Consequently, the forward rate curve has a shape similar to the standard shape of marginal cost curves. We see that the spot and forward rate curves start at the same point for zero time to maturity (the overnight rate); then while the spot rate is decreasing in the time to maturity the forward rate is below the spot rate. The forward rate curve cuts the spot rate curve in the latter's minimum, and when spot rates are increasing in the time to maturity the forward rate is above the spot rate.

The forward rate has a relatively complex shape on December 29, 1993, with a conspicuous kink for about three-month settlement. Therefore, the fit with the original Nelson and Siegel functional form is unsatisfactory, and the extended variant results in a much better fit. For details about the fit with the original form and for minimized price errors, see the Appendix.

Conclusions

A convenient method to estimate implied forward rates has been presented. The method has a precision much above what appears necessary for monetary policy analysis. It easily allows minimizing yield errors as well as price errors. The former may often be more relevant for monetary policy purposes. The method involves the use of a functional form that allows a considerable amount of flexibility. The functional form imposes a horizontal asymptote on the forward rate curve. This seems warranted for monetary policy purposes; there appears to be no reason why current expectations of short interest rates twenty years ahead should be any different from current expectations thirty years ahead. The horizontal asymptote avoids the problem with unstable estimates of long forward rates that is frequently observed with spline methods.

The forward rate curve contains the same information as the spot rate curve, but it presents the information in a way that makes it easier to interpret for monetary policy purposes. Thus the forward rate curve separates market expectations for the short, medium and long term more easily than the spot rate curve. Since monetary policy measures have effects with »long and variable lags«, looking beyond the short term is often necessary in monetary policy analysis. Using long spot rates or long bond yields instead of long forward rates can give a misleading impression, since long spot rates and long bond yields include expectations of interest rate movements in the short term. In long forward rates, expectations of interest rate movements in the short term have been filtered out.

The practical use of forward interest rates as monetary policy indicators is demonstrated, for instance, in the references given in footnote 1 (page 13). The interpretation of forward rates as expected future interest rates requires assumptions of negligible, constant, or predictable term premia. The interpretation of forward rates and the corresponding assumptions on various risk premia are discussed, for instance, in Svensson (1994a,b). Campbell (1995), Froot (1989), Hardouvelis (1994), Dahlquist (1995) and Gerlach and Smets (1995) provide recent estimates and discussion of term premia. Söderlind (1995) reports recent results on the use of forward rates as indicators of inflation expectations.

Appendix. Details on the estimation of forward rates

For a given trade date, let there be n coupon bonds $(c_j, m_j, y_j, p_j), j=1,\ldots,n$, where c_j, m_j, y_j and p_j denote, respectively, the coupon rate, the time to maturity, the observed yield to maturity and the observed price of bond j, which is assumed to have a face value of 100 units of domestic currency. (The bond prices are computed from the yields to maturity according to (3), or vice versa.) For a given parameter vector ,b, the estimated prices of the bonds $P_j(b)$, are computed with the discount function, $d(m;b)$, in (12) evaluating each coupon payment.

More precisely, for bonds with annual coupon payments, let τ_{jk}, $k=1, 2,\ldots, K_j$, denote the times to the coupon payments on bond j, where K_j is the number of coupon payments. In the special case when m_j is an integer, we simply have $\tau_{jk} = k$ and $K_j = m_j$. In the general case we have

$$\text{(A.1)} \qquad \begin{aligned} &\tau_{jk}=m_j-[m_j]+k-1 \text{ and} \\ &K_j = [m_j]+1, \end{aligned}$$

where $[m_j]$ denotes the largest integer that is strictly smaller than m_j. The estimated price of each bond, $P_j(b)$, is the present value of the bond when the coupon payments and the face value are priced with the discount function,

$$\text{(A.2)} \qquad \begin{aligned} &P_j(b) \equiv \sum_{k=1}^{K_j} c_j d(\tau_{jk};b) + 100 d(\tau_{jK_j};b), \\ &j=1,\ldots,n. \end{aligned}$$

For semi-annual coupon payments, that is, for Britain and the United States, these relations are accordingly modified (see for instance Fage (1986)).

When price errors are minimized, the observed price is assumed to differ from the estimated price by an error term, ε_j, (A.3) $\qquad p_j = P_j(b) + \varepsilon_j.$

The estimated prices are then fitted to the observed prices with Non-linear Least Squares, the General Method of Moments, or Maximum Likelihood. The estimates in this paper are Maximum Likelihood. The 95 per cent confidence intervals have been computed with the delta method and are heteroskedasticity-consistent.[9]

When yield errors are minimized, the estimated yield to maturity for bond j, $Y_j(b)$, is computed from the observed bond price $P_j(b)$ by solving (3). Although (3) is a non-linear higher-order equation with the same order as the number of coupon payments, it has only one real root and is easy to solve numerically, for instance with the standard Newton-Raphson algorithm (see Fage (1986)). The observed yield to maturity, y_j, is assumed to differ from the estimated yield to maturity by an error term,

(A.4) $\qquad y_j = Y_j(b) + \varepsilon_j,$

and the estimated yields to maturity are then fitted to the observed yields to maturity.

The estimation has been done with the restriction that the forward rate curve (and hence the spot rate curve) should start at the left end from the marginal lending rate or the overnight rate.

9. Let $\hat{b}$ and $\hat{\Sigma}$ denote the estimates of the parameter vector b and its covariance matrix, respectively. The delta method implies that for the purpose of computing confidence intervals for the forward rate, the estimated forward rate, $f(m;\hat{b})$, for a given settlement m is considered to be distributed as a Normal variable with mean $f(m;\hat{b})$ and covariance $\frac{\partial f(m;\hat{b})'}{\partial b}\hat{\Sigma}\frac{\partial f(m;\hat{b})}{\partial b}$ where $\frac{\partial f(m;\hat{b})}{\partial b}$ is the column vector of partial derivatives with respect to the parameters, etc.

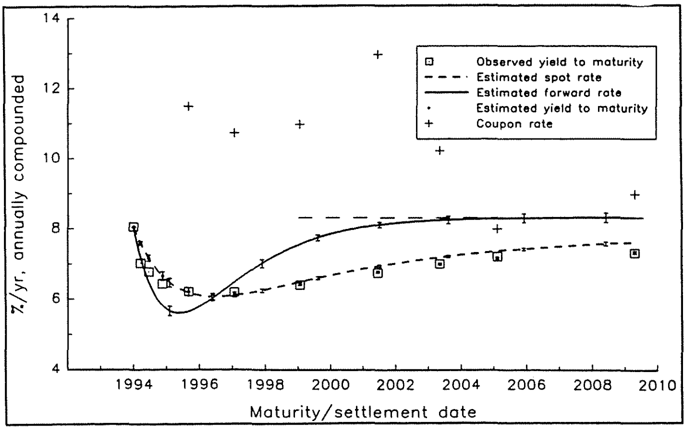
The difference between minimizing price and yield errors, and between the original and the extended Nelson and Siegel functional form, is illustrated in Figures 2 and Figures A1–A3.

Figure A1a shows the estimated spot and forward rate curves for December 29, 1993, when price errors are minimized with the original Nelson and Siegel functional form in (9). We see that the fit for the Treasury bill yields is unsatisfactory. *Figure A1b* shows estimated and observed prices. For the prices the fit is good. The estimated prices are close to the observed prices, and the error bars that denote 95 per cent confidence intervals are hardly visible in the figure. The root mean squared price error (RMSPE) is 0.18 kronor (for bonds with face value 100 kronor), whereas the root mean squared yield error (RMSYE) is 0.22 percentage points. The maximum absolute yield error is 0.57 percentage points and occurs for the three-month Treasury bill.

Figure A2a shows the estimated spot and forward rate curves for the same trade date, when instead yield errors are minimized, still with the original Nelson and Siegel functional form. The RMSYE has fallen to 0.16 percentage points. The maximum absolute yield error has fallen to 0.38 percentage points and still occurs for the three-month Treasury bill. The fit for the yields is still hardly satisfactory. The confidence intervals for the forward rates are rather large. The RMSPE has increased to 0.51 kronor. We see in *Figure A2b* that the fit for prices is worse than in *Figure A1b*, and that the confidence intervals are larger.

Figure 2 shows the estimated spot and forward rates when yield errors are minimized and when the extended Nelson and Siegel functional form is used. *Figure A3* shows the corresponding observed and

FIGURE A I A. *Minimized Price Errors, Original N&S. Yields to Maturity, Spot and Forward Rates, Sweden, 29 December 1993*
95 % confidence interval



estimated prices. The fit is quite good for both yields and prices. The RMSYE is only 0.03 per centage points, and the RMSPE is 0.16 kronor. The maximum absolute yield error is 0.05 per centage points and occurs for the twelve-month Treasury bill. It is evident from Figure 1 that the second U-shape (the thin solid line) allows a much better fit at the short end of the spot rate curve.

The term structure on December 29 is relatively complex. In many cases the original Nelson and Siegel functional form with minimized yield errors gives a satis-

FIGURE A I B. *Minimized Price Errors, Original N&S. Observed and Estimated Bond Prices, Coupons, Sweden, 29 December 1993*
95 % confidence interval

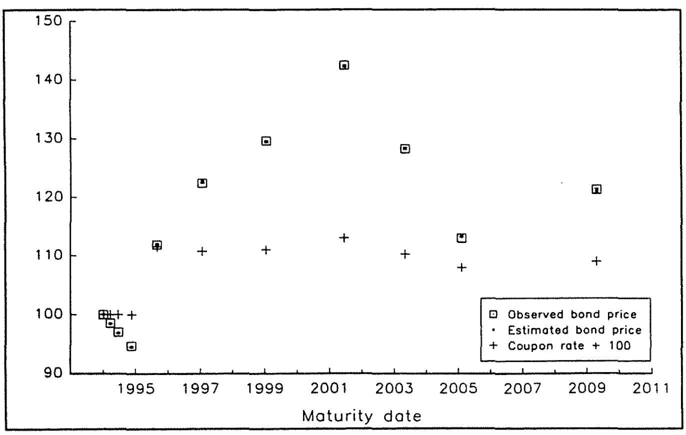
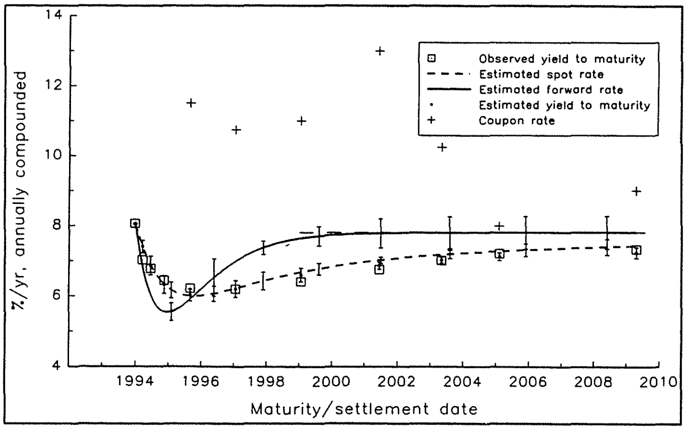


FIGURE A2A. *Minimized Yield Errors, Original N&S. Yields to Maturity, Spot and Forward Rates, Sweden, 29 December 1993*
95 % confidence interval



factory result. In some cases the difference between the estimates for minimized yield errors and minimized price errors is small.

When the extended functional form (10) is used, it is obvious that perfect multicollinearity results if $\tau_1=\tau_2$. Then only the sum $\beta_2+\beta_3$ can be determined, not the individual components β_2 and β_3. In this case the two hump-shapes are located on top of each other, the model is over-parameterized, and the variance-covariance matrix cannot be computed. Thus, it is important that appropriate initial parameter values with $\tau_1\neq\tau_2$ are given. If the

FIGURE A2B. *Minimized Yield Errors, Original N&S. Observed and Estimated Bond Prices, Coupons, Sweden, 29 December 1993*
95 % confidence interval

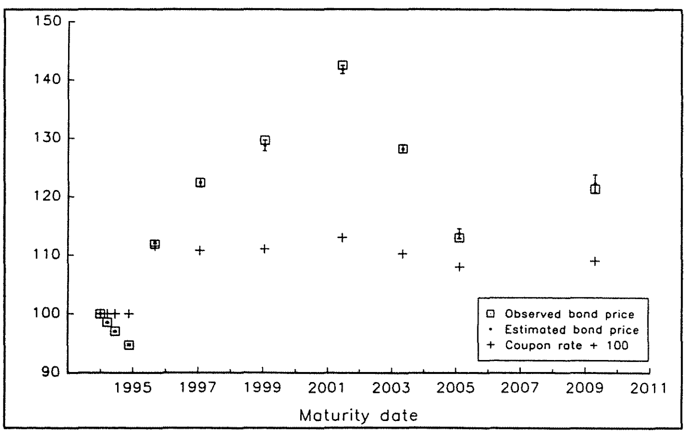
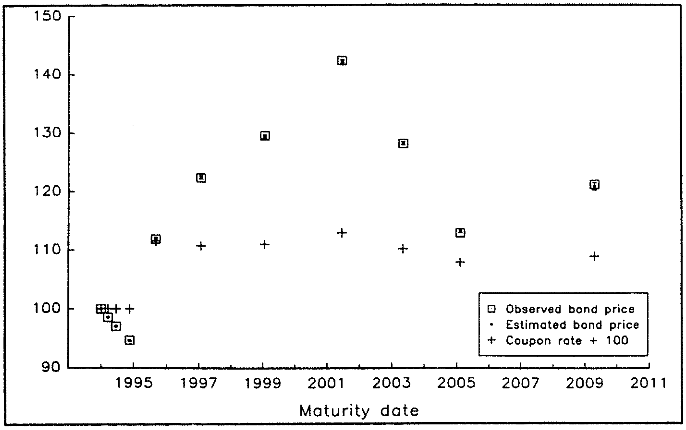


FIGURE A3. *Minimized Yield Errors, Extended N&S. Observed and Estimated Bond Prices, Coupons, Sweden, 29 December 1993*
95 % confidence interval



estimation converges to $\tau_1 = \tau_2$, the original Nelson and Siegel functional form is appropriate for that observation. One way to avoid τ_1 and τ_2 becoming equal is to add a punishment term to the likelihood function.[10]

References

Bank of England (1994), *Inflation Report*, August 1994.

Bank for International Settlements (1993), *63rd Annual Report*, Basle.

Campbell, John Y. (1995), »Some Lessons from the Yield Curve,« *Journal of Economic Perspectives*, forthcoming.

Dahlquist, Magnus (1995), »Conditional Forward Term Premia: Some International Evidence«, Working Paper, IIES.

Dahlquist, Magnus, and Lars E.O. Svensson (1995), »Estimation of the Term Structure of Interest Rates with Simple and Complex Functional Forms: Nelson & Siegel vs. Longstaff & Schwartz,« *Scandinavian Journal of Economics*, forthcoming.

Deacon, Mark, and Andrew Derry (1994), »Estimating Market Interest Rate and Inflation Expectations from the Prices of UK Government Bonds,« *Bank of England Quarterly Review*, August 1994, 232–240.

Fage, Paul (1986), *Yield Calculations*, Credit Suisse First Boston, London.

Fama, Eugene F., and Robert R. Bliss (1987), »The Information in Long-Maturity Forward Rates,« *American Economic Review* 77, 608–692.

Fisher, Mark E., Douglas Nychka and David Zervos (1994), »Fitting the Term Structure of Interest Rates with Smoothing Splines,« Working Paper, Board of Governors of the Federal Reserve System.

Froot, Kenneth A. (1989), »New Hope for the Expectations Hypothesis of the Term Structure of Interest Rates,« *Journal of Finance* 44, 283–305.

10. This was suggested by Jon Faust.

Gerlach, Stefan and Smets, Frank (1995), »The Term Structure of Euro-Rates: Some Evidence in Support of the Expectations Hypothesis,« Working Paper, BIS.

Hardouvelis, Gikas A. (1994), »The Term Structure Spread and Future Changes in Long and Short Rates in the G7 Countries: Is There a Puzzle?« *Journal of Monetary Economics* 33, 255–283.

Longstaff, Francis A., and Eduardo S. Schwartz (1992), »Interest Rate Volatility and the Term Structure: A Two-Factor General Equilibrium Model,« *Journal of Finance* 47, 1259–1282.

McCulloch, J. Huston (1971), »Measuring the Term Structure of Interest Rates,« *Journal of Business* 44, 19–31.

McCulloch, J. Huston (1975), »An Estimate of the Liquidity Premium,« *Journal of Political Economy* 83, 62–63.

Nelson, Charles R., and Andrew F. Siegel (1987), »Parsimonious Modeling of Yield Curves,« *Journal of Business* 60, 473–489.

Reinhart, Vincent, and Leora Klapper (1992), »Understanding the Simple Algebra of Forward Rates,« Working Paper, Board of Governors of the Federal Reserve System.

Shea, Gary S. (1984), »Pitfalls in Smoothing Interest Rate Term Structure Data: Equilibrium Models and Spline Approximations,« *Journal of Financial and Quantitative Analysis* 19, 253–269.

Shiller, Robert J. (1990), »The Term Structure of Interest Rates,« Chapt. 13 in Friedman, Ben M., and Frank H. Hahn (eds), *Handbook of Monetary Economics, Volume I*, North-Holland, Amsterdam.

Söderlind, Paul (1995), »Forward Interest Rates as Indications of Inflation Expectations,« IIES Seminar Paper No. 594.

Svensson, Lars E.O. (1993), »Estimation of Forward Interest Rates,« *Penning- & Valutapolitik* 1993:3, 33–43.

Svensson, Lars E.O. (1994a), »Estimating and Interpreting Forward Interest Rates: Sweden 1992–1994,« Working Paper No. WP/94/114, International Monetary Fund, Washington, D.C.

Svensson, Lars E.O. (1994b), »Monetary Policy with Flexible Exchange Rates and Forward Interest Rates as Indicators,« *Cahiers Economiques et Monétaires* No. 43, Banque de France, 305–332.

Sveriges Riksbank (1993a), *Inflation and Inflation Expectations*, October 1993.

Sveriges Riksbank (1993b), *Monetary Policy Indicators*, June 1993.