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Solving a System of Linear Difference Equations with Nonpredetermined Variables: Lecture Notes

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Consider the system

$$H\begin{bmatrix} y_{1,t+1} \\ \mathrm{E}_t y_{2,t+1} \end{bmatrix} = A\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix} \tag{0.1}$$

for $t \geq 0$, where y_{1t} is an n_1-vector of predetermined variables and y_{10} is given, y_{2t} is an n_2-vector of nonpredetermined variables, ε_{t+1} is an iid random n_1-vector with zero mean. The real matrices A and H are $n \times n$, where $n \equiv n_1 + n_2$. Take expectations conditional on information in period t and write the system as

$$H\begin{bmatrix} \mathrm{E}_t y_{1,t+1} \\ \mathrm{E}_t y_{2,t+1} \end{bmatrix} = A\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} \tag{0.2}$$

Following Klein [4], Sims [5], and Söderlind [6], use the generalized Schur decomposition of C. This decomposition results in the square possibly complex matrices Q, S, T, and Z such that

$$A = Q'TZ', \tag{0.3}$$
$$H = Q'SZ', \tag{0.4}$$

where Z' for a complex matrix denotes the complex conjugate transpose of Z (the transpose of the complex conjugate of Z). The matrices Q and Z are unitary ($Q'Q = Z'Z = I$), and S and T are upper triangular (see Golub and van Loan [2]). Furthermore, the decomposition is sorted according to ascending modulus of the generalized eigenvalues, so $|\lambda_j| \geq |\lambda_k|$ for $j \geq k$. [1] (The generalized eigenvalues are the ratios of the diagonal elements of T and S, $\lambda_j = t_{jj}/s_{jj}$ ($j = 1, ..., n$). A generalized eigenvalue is infinity if $t_{jj} \neq 0$ and $s_{jj} = 0$ and zero if $t_{jj} = 0$ and $s_{jj} \neq 0$.)

Assume the *saddle-point property* (Blanchard and Kahn [1]): The number of generalized eigenvalues with modulus larger than unity (the unstable eigenvalues) equals the number of nonpredetermined variables. Thus, I assume that $|\lambda_j| > 1$ for $n_1 + 1 \leq j \leq n_1 + n_2$ and $|\lambda_j| < 1$ for

[1] The sorting of the eigenvalues is often done by two programs written by Sims and available at www.princeton.edu/∼sims, Qzdiv and Qzswitch.

$1 \leq j \leq n_1$.(for an exogenous predetermined variable with a unit root, I can actually allow $|\lambda_j| = 1$ for some $1 \leq j \leq n_1$).

Define

$$\begin{bmatrix} \tilde{y}_{1t} \\ \tilde{y}_{2t} \end{bmatrix} \equiv Z' \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix}. \tag{0.5}$$

I can interpret $\tilde{y}_{1t}$ as a complex vector of n_1 transformed predetermined variables and $\tilde{y}_{2t}$ as a complex vector of n_2 transformed non-predetermined variables. Premultiply the system (0.2) by Q and use (0.3)-(0.5) to write it as

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} \mathrm{E}_t\tilde{y}_{1,t+1} \\ \mathrm{E}_t\tilde{y}_{2,t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} \tilde{y}_{1t} \\ \tilde{y}_{2t} \end{bmatrix}, \tag{0.6}$$

where S and T have been partitioned conformably with $\tilde{y}_{1t}$ and $\tilde{y}_{2t}$.

Consider the lower block of (0.6),

$$S_{22}\,\mathrm{E}_t\tilde{y}_{2,t+1} = T_{22}\,\tilde{y}_{2t}. \tag{0.7}$$

Since the diagonal terms of S_{22} and T_{22} (s_{jj} and t_{jj} for $n_1 + 1 \leq j \leq n_1 + n_2$) satisfy $|t_{jj}/s_{jj}| > 1$, the diagonal terms of T_{22} are nonzero, the determinant of T_{22} is nonzero, and T_{22} is invertible. Note that S_{22} may not be invertible. I can then solve for $\tilde{y}_{2t}$ as

$$\tilde{y}_{2t} = J\mathrm{E}_t\tilde{y}_{2,t+1} = 0, \tag{0.8}$$

where the complex matrix J is given by

$$J \equiv T_{22}^{-1}S_{22}. \tag{0.9}$$

I exploit that the modulus of the diagonal terms of $T_{22}^{-1}S_{22}$ is less than one. I also assume that $\mathrm{E}_t\tilde{y}_{2,t+\tau}$ are sufficiently bounded. Then $J^\tau \mathrm{E}_t\tilde{y}_{2,t+\tau} \to 0$ when $\tau \to \infty$. Note that J may not be invertible, since S_{22} may not be invertible.

I have, by (0.5),

$$y_{1t} = Z_{11}\tilde{y}_{1t}, \tag{0.10}$$

$$y_{2t} = Z_{21}\tilde{y}_{1t}, \tag{0.11}$$

where

$$Z \equiv \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \tag{0.12}$$

is partitioned conformably with y_{1t} and y_{2t}. Under the assumption of the saddle-point property, Z_{11} is square. I furthermore assume that Z_{11} is *invertible*. Then I can solve for $\tilde{y}_{1t}$ in (0.10),

$$\tilde{y}_{1t} = Z_{11}^{-1}y_{1t}, \tag{0.13}$$

and use this in (0.11) to get

$$y_{2t} = Fy_{1t}, \tag{0.14}$$

where the real $n_2 \times n_1$ matrix F is given by

$$F \equiv Z_{21}Z_{11}^{-1}. \tag{0.15}$$

It remains to find a solution for $y_{1,t+1}$. By (0.8), the upper block of (0.6) is

$$S_{11}\mathrm{E}_t\tilde{y}_{1,t+1} = T_{11}\tilde{y}_{1t}.$$

Since the diagonal terms of S_{11} and T_{11} satisfy $|t_{jj}/s_{jj}| < 1$, all diagonal terms of S_{11} must be nonzero, so the determinant of S_{11} is nonzero, and S_{11} is invertible. I can then solve for $\mathrm{E}_t\tilde{y}_{1,t+1}$ as

$$\mathrm{E}_t\tilde{y}_{1,t+1} = S_{11}^{-1}T_{11}\tilde{y}_{1t}.$$

By (0.10),

$$\begin{aligned} \mathrm{E}_t y_{1,t+1} &= Z_{11}\mathrm{E}_t\tilde{y}_{1,t+1} \\ &= Z_{11}S_{11}^{-1}T_{11}\tilde{y}_{1t} \\ &= Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1}y_{1t} \end{aligned} \tag{0.16}$$

where I have used (0.13).

It follows that I can write the solution as

$$y_{1,t+1} = My_{1t} + \varepsilon_{t+1}, \tag{0.17}$$

where the real matrix M is given by

$$M \equiv Z_{11}S_{11}^{-1}T_{11}Z_{11}^{-1} \tag{0.18}$$

Thus, the solution to the system (0.1) is given by (0.14) and (0.17) for $t \geq 0$.

References

[1] Blanchard, Olivier J., and Charles M. Kahn (1980), "The Solution of Linear Difference Models under Rational Expectations," *Econometrica* 48, 1305–1311.

[2] Golub, Gene H., and Charles F. van Loan (1989), *Matrix Computation*, 2nd ed., Johns Hopkins University Press, Baltimore, MD.

[3] Harville, David A. (1997), *Matrix Algebra From a Statistician's Perspective*, Springer-Verlag, New York.

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[5] Sims, Christopher A. (2000), "Solving Linear Rational Expectations Models," working paper, www.princeton.edu/∼sims.

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